Tues, 2-12-19

4. Spectrum

1. Lightning talk
2. Self assessment on A3
3. **The frequency domain**
4. Sinusoidal model of a time series
5. Spectral analysis

Read notes_4.pdf for next class

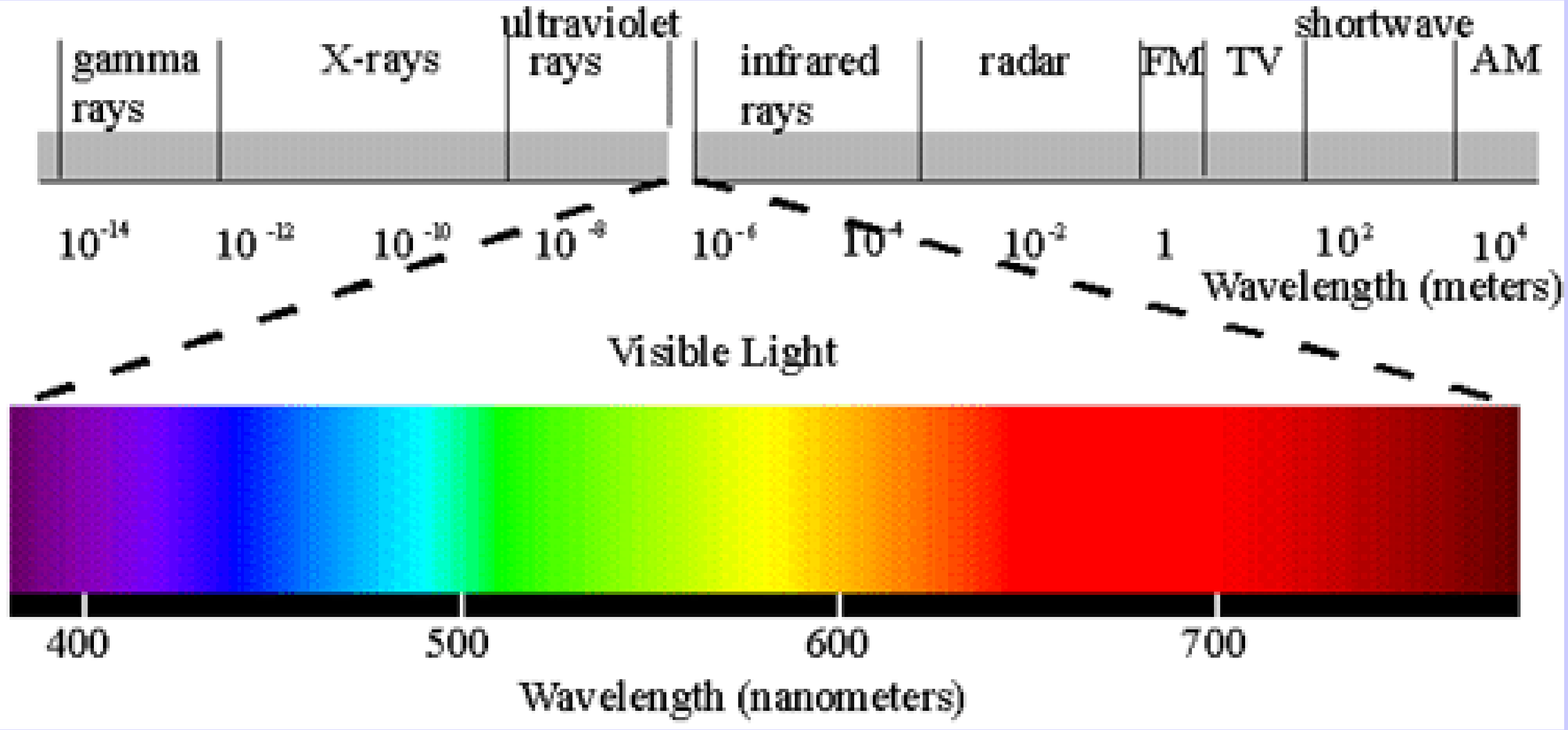
A3 Feedback

1. Download A3x.pdf from D2L
2. Automatic points, for running assignment and having uploaded by due time, is already marked in parentheses at top of first page
3. Each assignment has maximum possible 10 points; if you make no deductions, score is 10/10
4. Open your copy of the same assignment pdf you uploaded
5. In Acrobat Reader, using “Add text box,” mark in right margin for deductions only, with deduction and segment reference : (eg., -0.5 A)
6. At top of your pdf, mark grade like this : 9.5/10
7. If necessary, put any comments at top near the grade
8. Upload your self-graded pdf to folder A3_graded in D2L

Goal: understand the spectrum

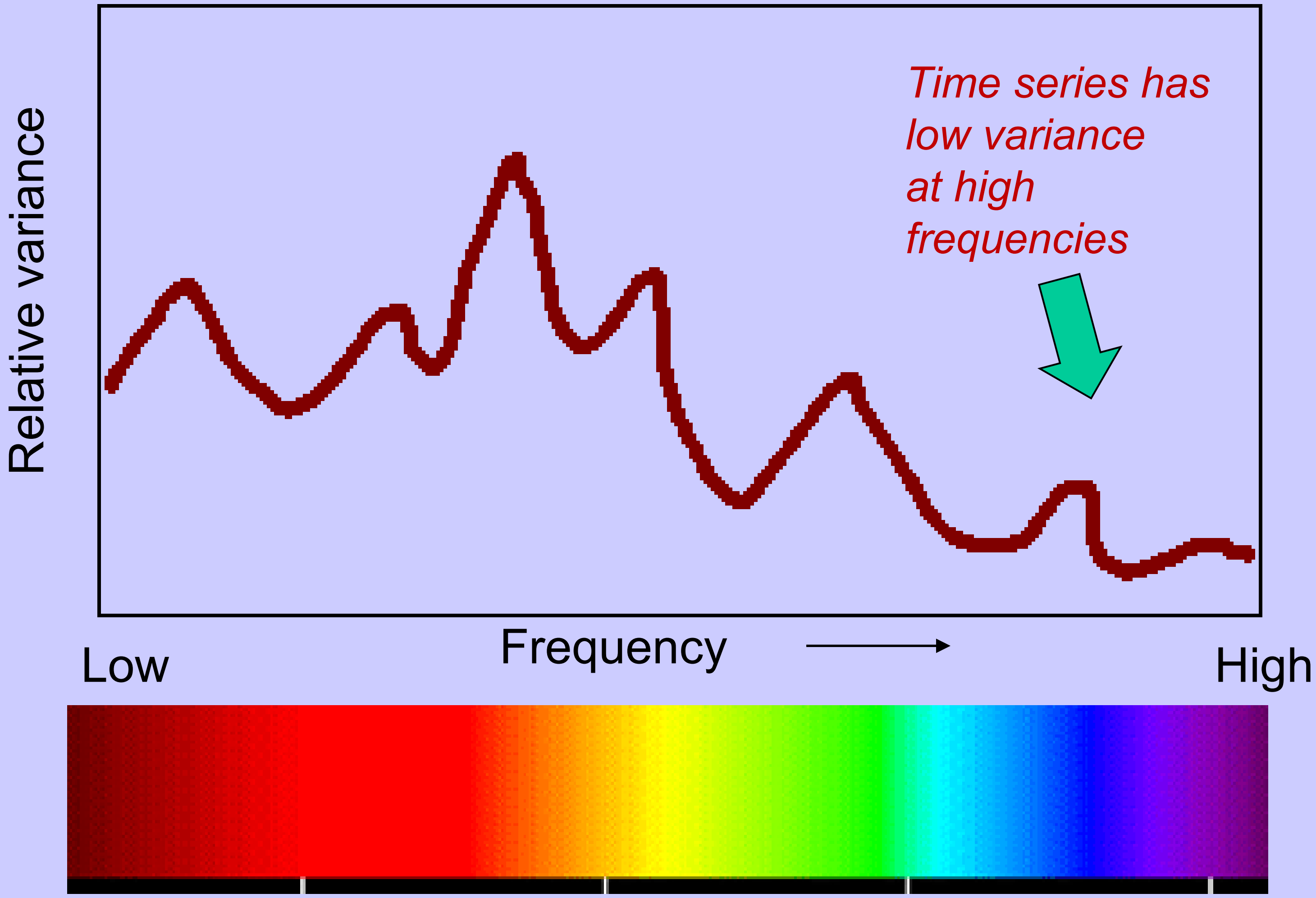
The spectrum of a time series is the variance of the series as a function of frequency

Analogy to optical spectrum



- *Contributions of different wavelengths or frequencies to the energy of a given light source*
- *Contributions of different wavelengths or frequencies to the variance of a time series*

Frequency domain

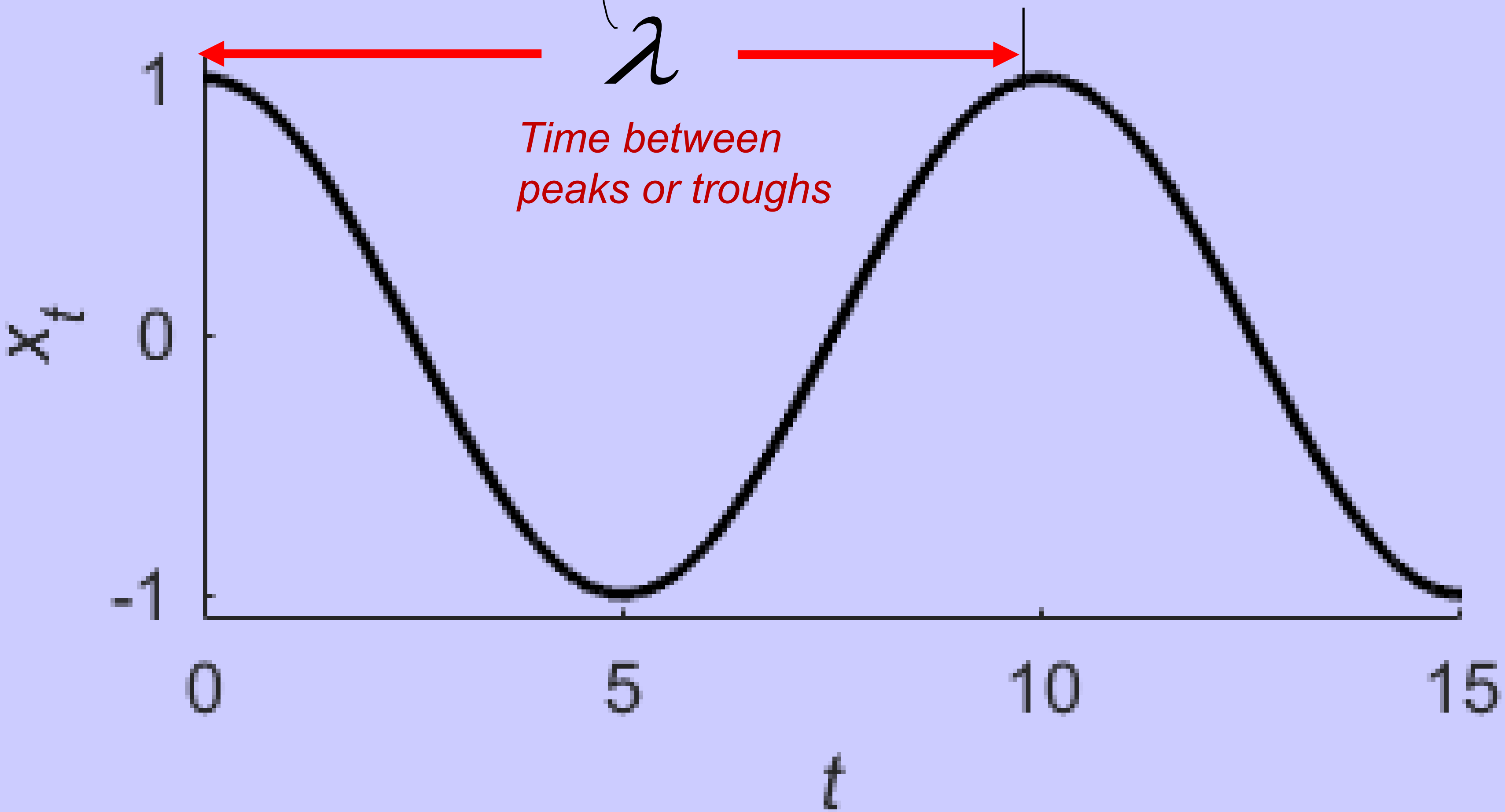


- Time domain: data at different times

- Frequency domain: variance at different wavelengths

Some terminology, using time series consisting of a
Single cosine wave…

wavelength, or period

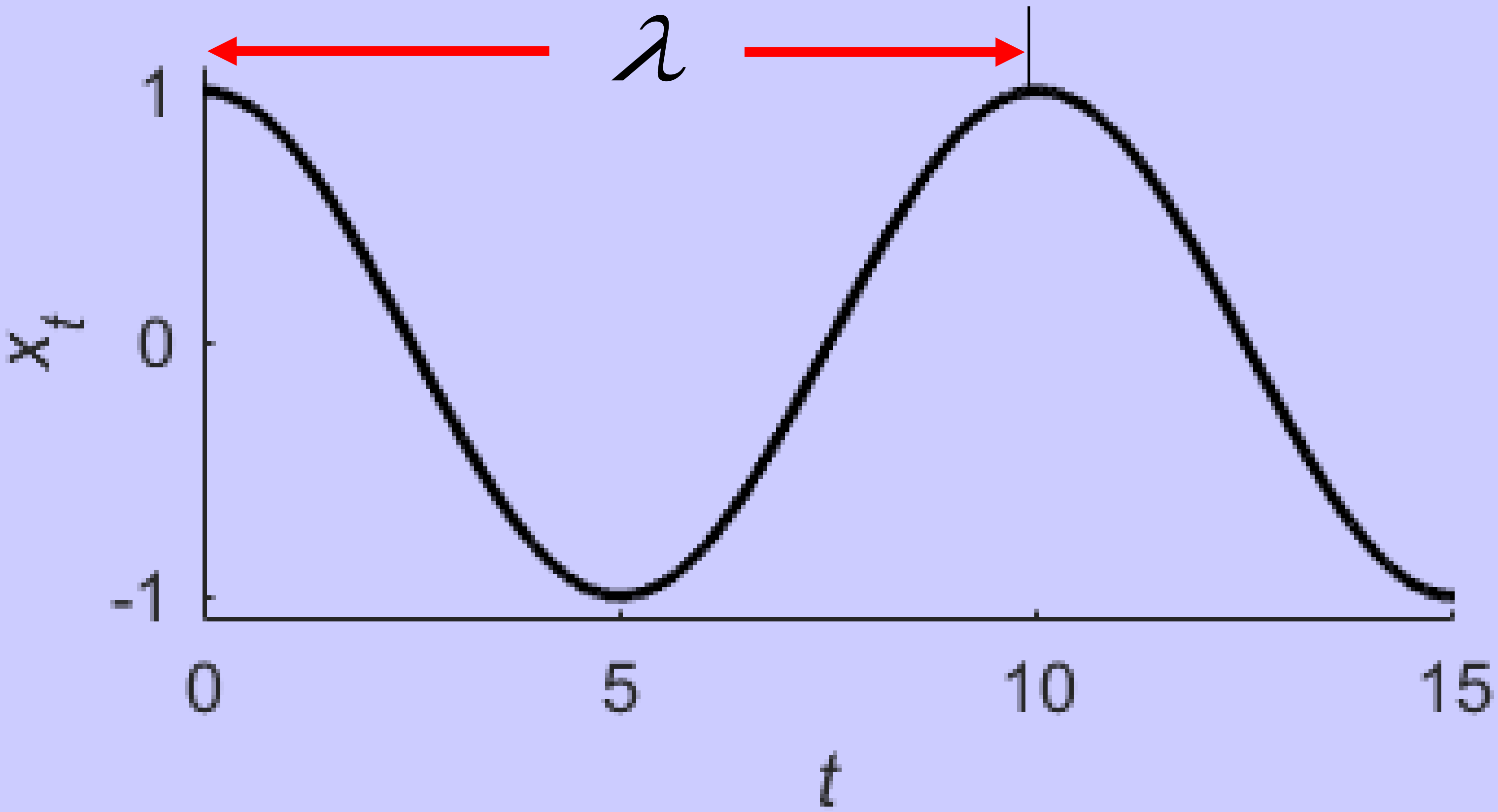


frequency

Frequency: $f = \dfrac{1}{\lambda}$ # cycles per time unit

Angular frequency: $\omega = 2\pi f$ # radians per time unit

2π radians $= 360^\circ =$ full cycle

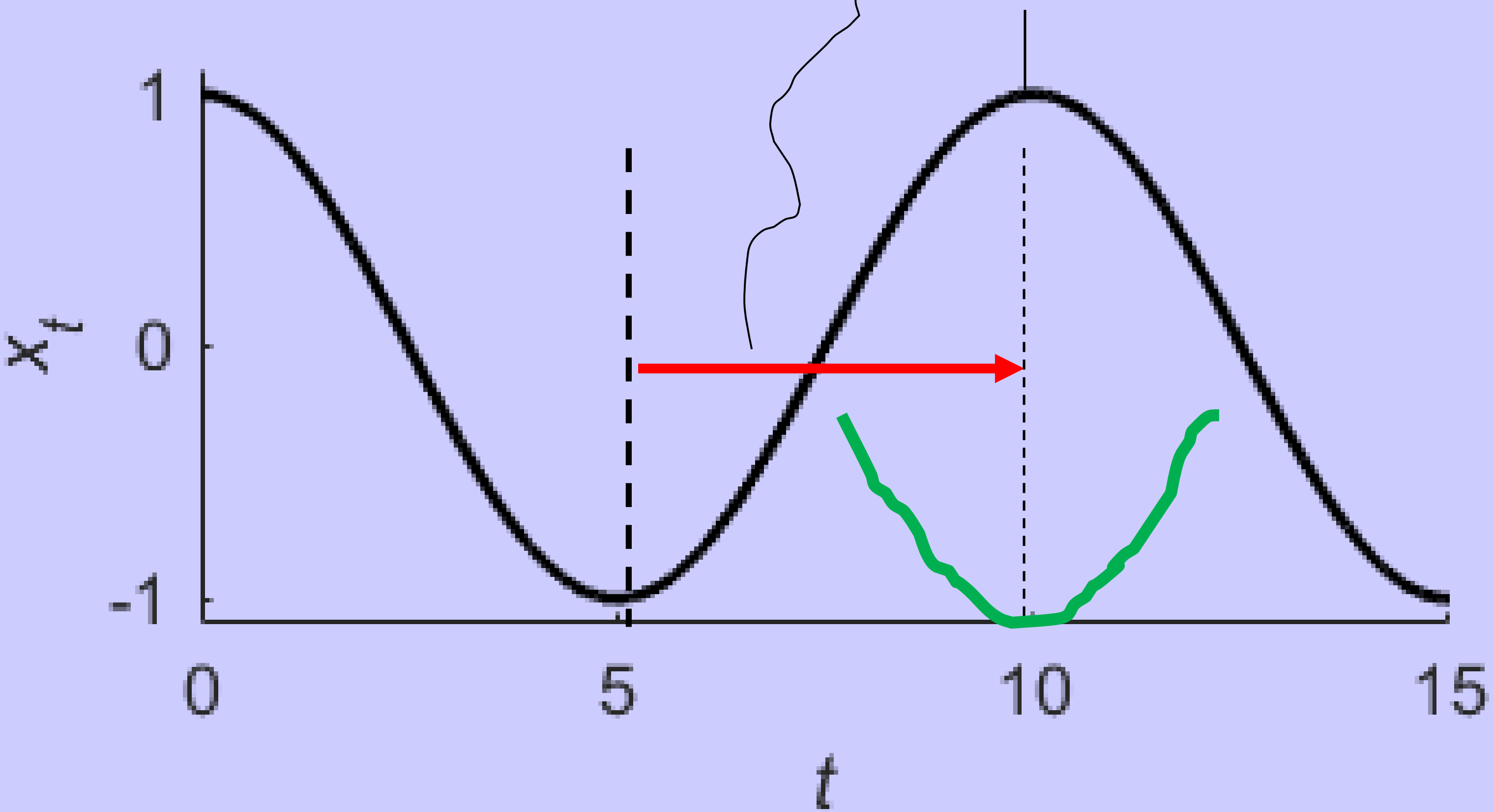


phase

Phase: ϕ

Time offset from arbitrary origin

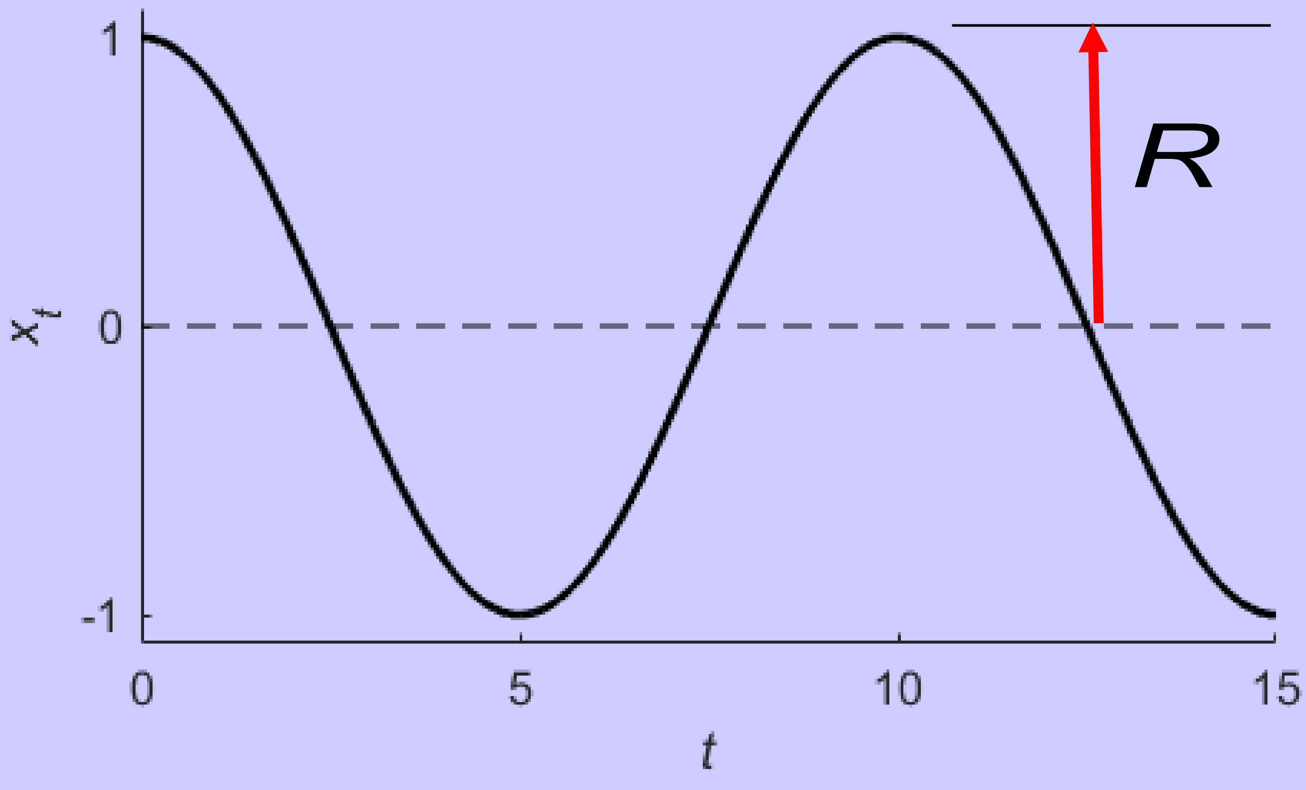
Green "trough" 180° out of phase with black trough



amplitude and variance

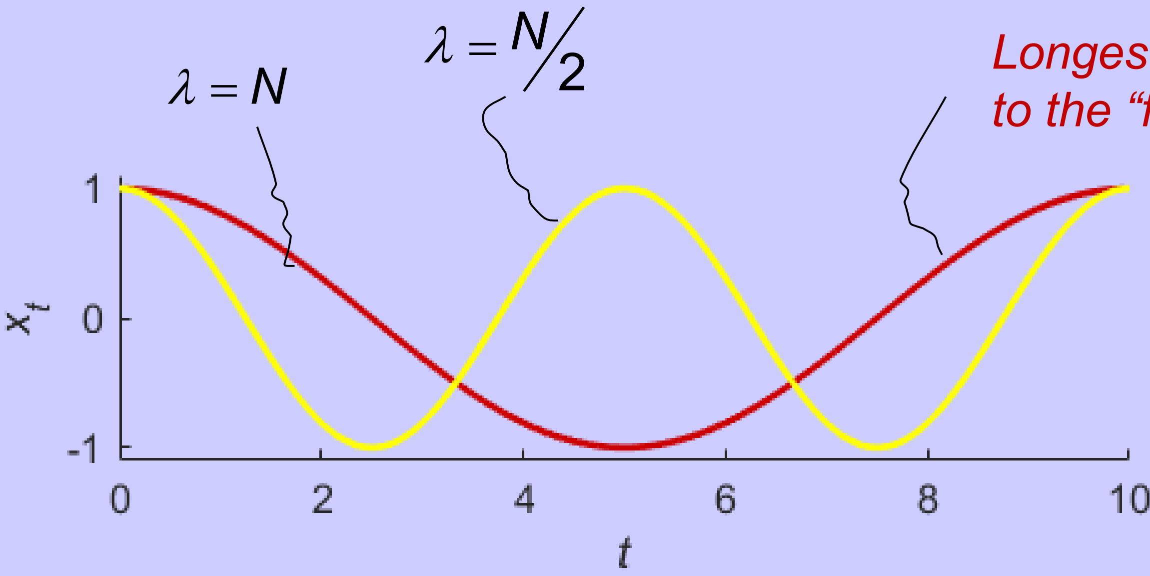
x_t : sinusoid with amplitude R

$$\mathrm{var}(x_t) = \frac{R^2}{2}$$



standard frequencies of a time series

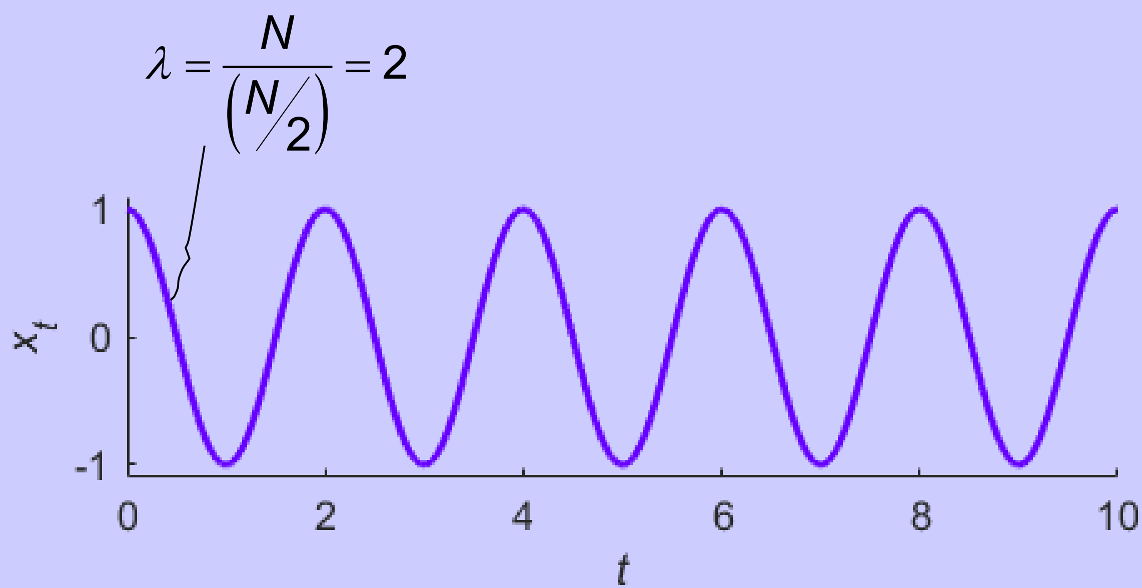
Longest wavelength corresponds to the "fundamental" frequency



- Time series of length N has N/2 standard frequencies
- These standard frequencies are at

$$f = 1/N, 2/N, 3/N, \dots \frac{(N/2)}{N}, \text{ or}$$

$$\lambda = N, \; N/2, N/3, \dots 2$$



Nyquist frequency

$$f = \frac{(N/2)}{N} = 0.5,$$

$$\lambda = \frac{1}{f} = 2$$

Sinusoidal model of a time series
Time series as sum of sinusoids

- Time series $x_t, \quad t = 1, N$
- Can view as sum of $N/2$ sinusoids
- These are at the standard frequencies

$$f = {}^{1}\!/_{N}, {}^{2}\!/_{N}, {}^{3}\!/_{N}, \ldots, {}^{\left({}^{N}\!/_{2}\right)}\!/_{N}$$

Time series as sum of sinusoids
mathematical model

$$X_t = \mu + \sum_{j=1}^{[N/2]} \left[A_j \cos\left(2\pi f_j t\right) + B_j \sin\left(2\pi f_j t\right) \right], \quad t=1,2,\ldots,N$$

Fourier, or standard, frequencies

$$f_j \equiv j/N, \qquad 1 \le j \le [N/2]$$

where N is the sample size

Variance at the standard frequencies

$$E\left\{A_j^{\,2}\right\} = E\left\{B_j^{\,2}\right\} = \sigma_j^{\,2}$$

Variance at *j*th standard frequency is proportional to squared amplitude of sinusoidal component at that frequency.

Total variance for sinusoidal model

$$\sigma^2 = E\left\{\left(X_t - \mu\right)^2\right\} = \sum_{j=1}^{[N/2]} \sigma_j^{\,2}$$

Total variance of series is sum of variance contributions at the *N/2* standard frequencies.

Definition of spectrum in terms of sinusoidal model

$$S_j^{\,2} \equiv \sigma_j^{\,2}, \qquad 1 \le j \le [N/2]$$

The spectrum at standard frequency *j* is defined as the contributed variance at that frequency. The spectrum summed over all standard frequencies therefore equals the total variance of the series.

An early application: harmonic analysis

- Assume sinusoidal model applies exactly
- Compute sinusoidal components at standard frequencies
- Interpret components (e.g., importance as inferred from variance accounted for)

Spectral Analysis

- View the time series as short random sample from infinitely long series; a single realization of a process
- Acknowledge that random sampling fluctuations can produce spurious peaks in the computed periodogram of the short sample
- Using the sample, estimate the spectrum of this infinitely long series, explicitly accounting for sampling variability

General definition of spectrum

Spectral distribution function:

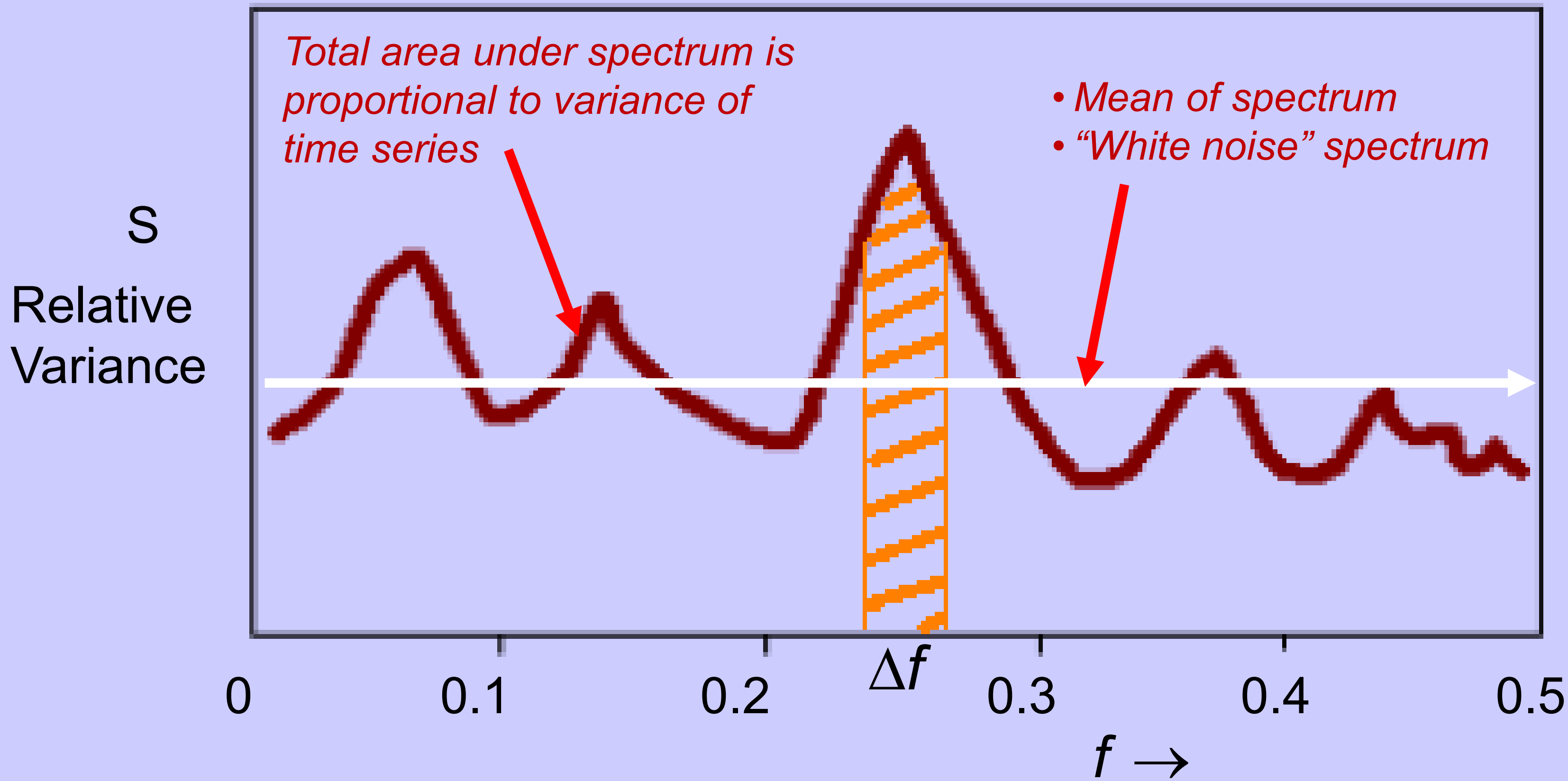
$F(\omega) =$ contribution to the variance of the series which is accounted for by frequencies in the range $(0,\omega)$

Spectral density function (spectrum):

$$f(\omega) = \frac{\mathrm{d}F(\omega)}{d\omega} \quad \equiv \text{(power) spectral density function}$$

The spectrum is the derivative of the spectral distribution function with respect to frequency. A point on the spectrum therefore represents the "variance per unit of frequency" at a specific frequency. If $\mathrm{d}\omega$ is an increment of frequency, the product $f(\omega)\mathrm{d}\omega$ is the contribution to the variance from the frequency range $(\omega,\omega+\mathrm{d}\omega)$.

Spectral density function



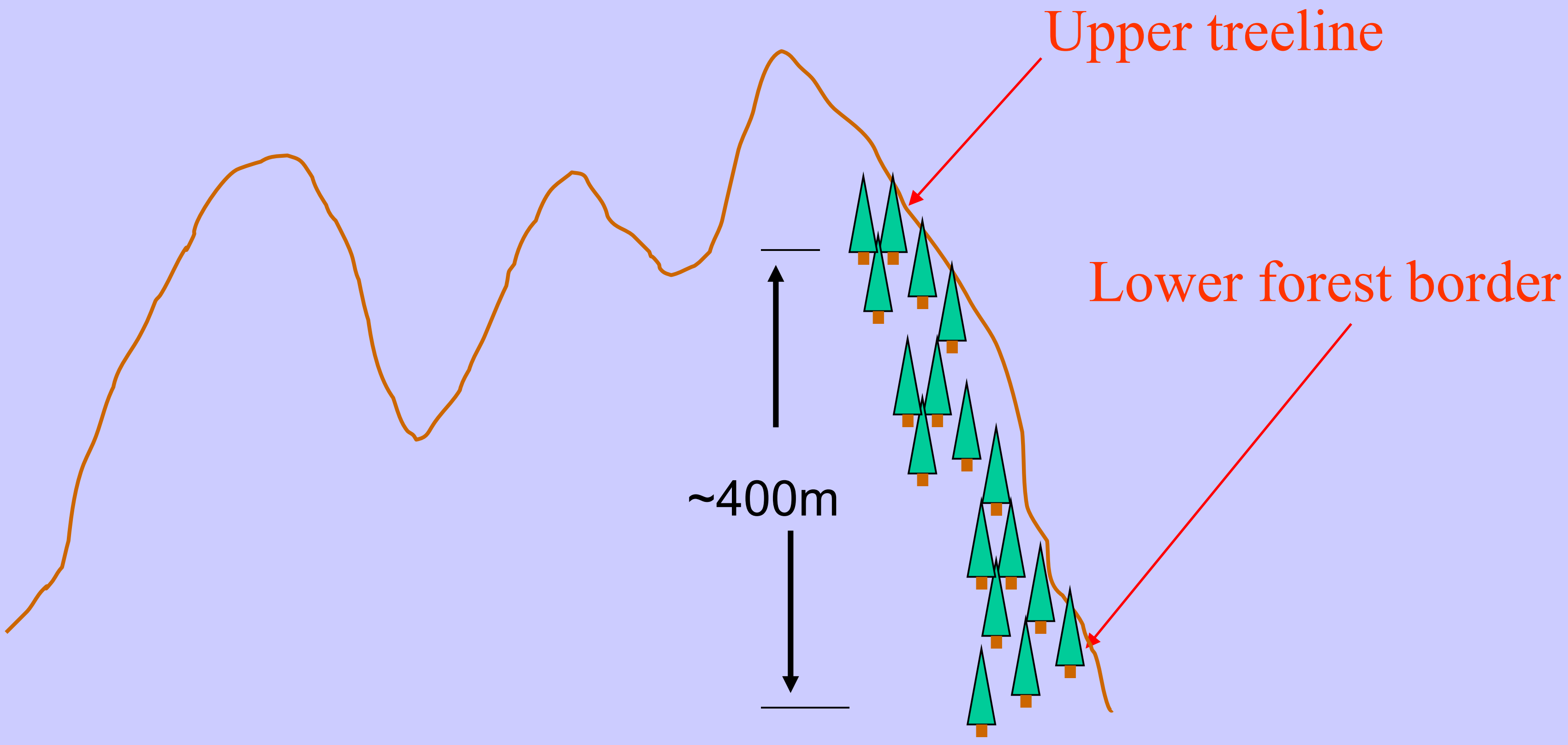
$A =$ hatched area

$A \propto$ variance in frequency range Δf

Why study the spectrum?

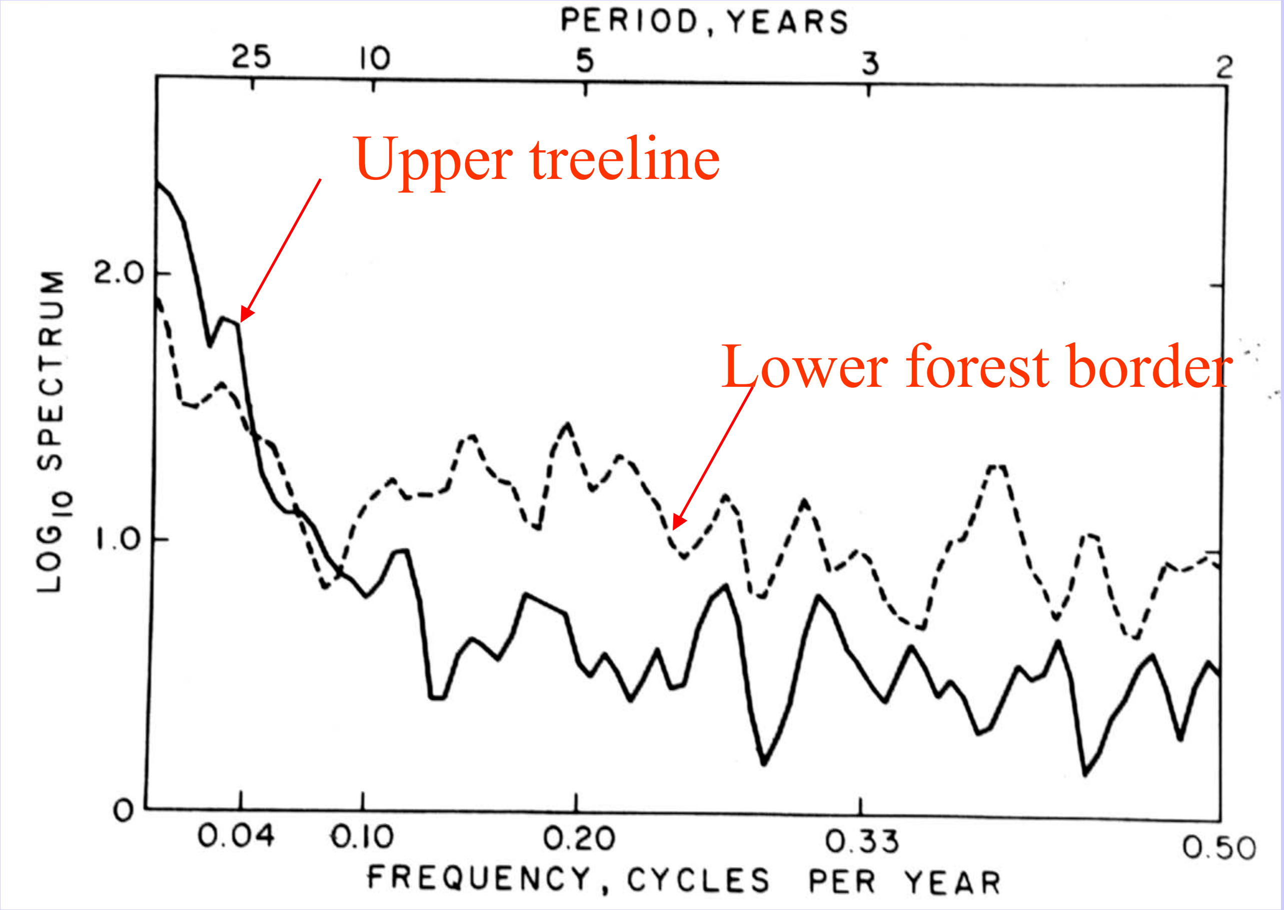
- Describe important timescales of variability
- Gain insight to biology or physics of system
- Forecast

Frequency Dependent Relationships Bristlecone Pine



LaMarche (1974)

Frequency Dependent Relationships



LaMarche (1974)