Tues, 2-26-19

Spectral Analysis: Smoothed-Periodogram Method

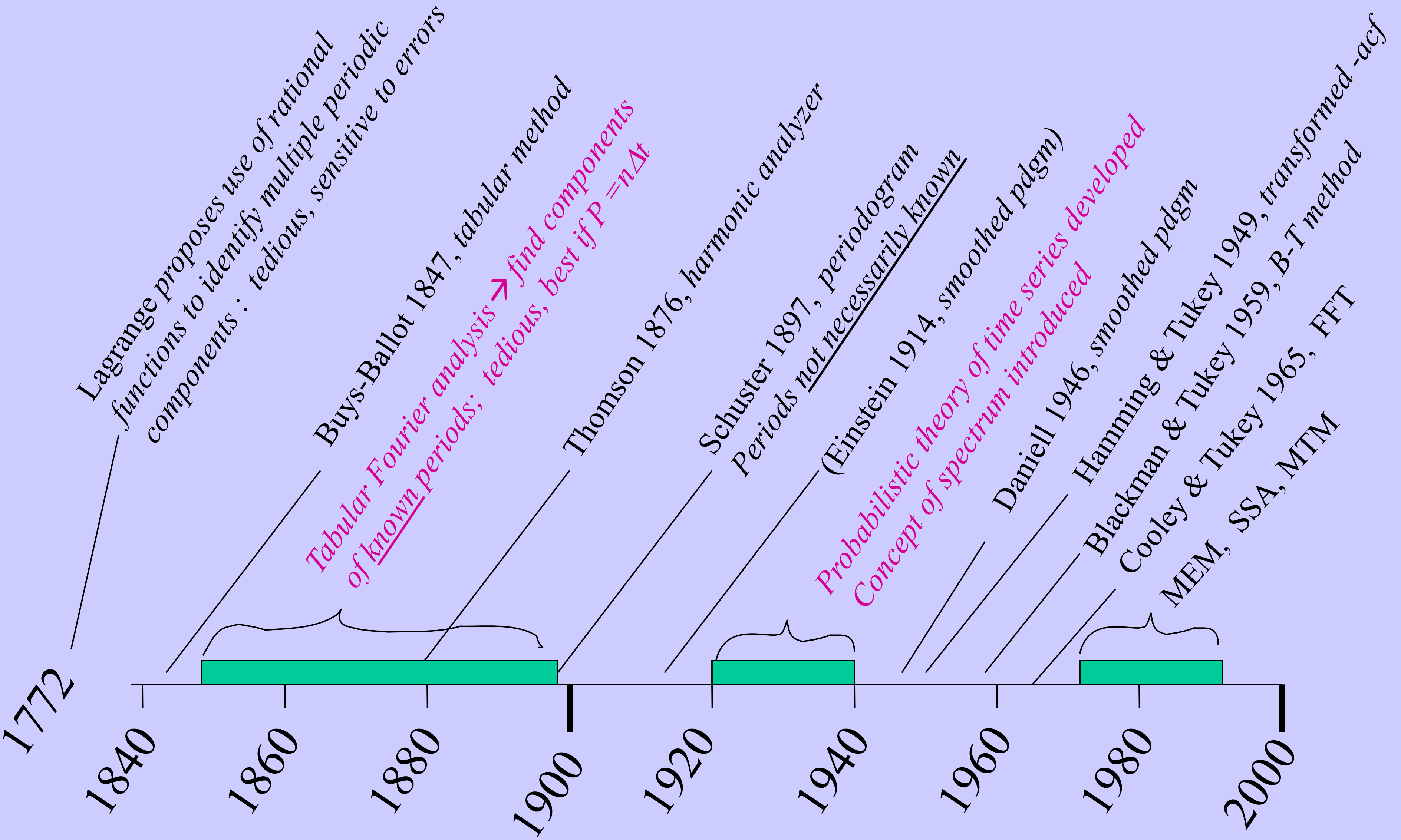
1. Lightning talk
2. Self assessment on A5
3. Historical context
4. Raw periodogram
5. Smoothed periodogram
6. Significance of spectral peaks

Read notes_6.pdf

A5 Feedback

1. Download A5x.pdf from D2L
2. Automatic points, for running assignment and having uploaded by due time, is already marked in parentheses at top of first page
3. Each assignment has maximum possible 10 points; if you make no deductions, score is 10/10
4. A5x is color coded for points; purple=1; yellow=0.5; blue=0.5
5. Open your copy of the same assignment pdf you uploaded
6. In Acrobat Reader, using “Add text box,” mark in right margin for deductions only, with deduction and segment reference : (eg., -0.5 A); round to tenths in deductions (e.g., no -0.25)
7. At top of your pdf, mark grade like this : 9.5/10
8. If necessary, put any comments at top near the grade
9. Upload your self-graded pdf to folder A5_graded in D2L

Historical Context



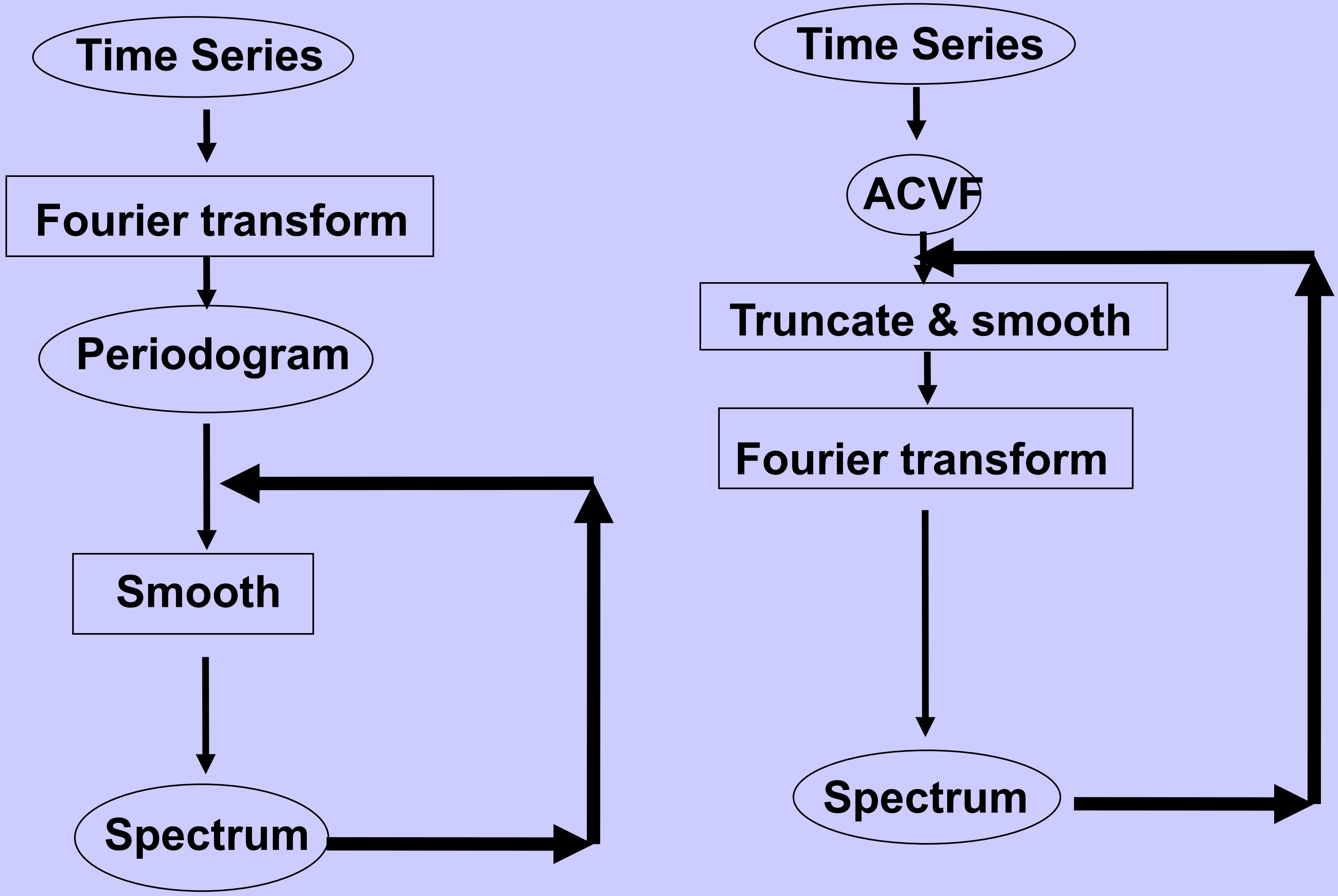
Source: Bloomfield 2000

Alternative Spectral Methods

1. *Blackman-Tukey
2. *Smoothed periodogram
3. Welch's
4. Multi-taper (MTM)
5. Singular spectrum analysis (SSA)
6. Maximum entropy (MEM)

* Used in class scripts

Smoothed Periodogram vs Blackman-Tukey

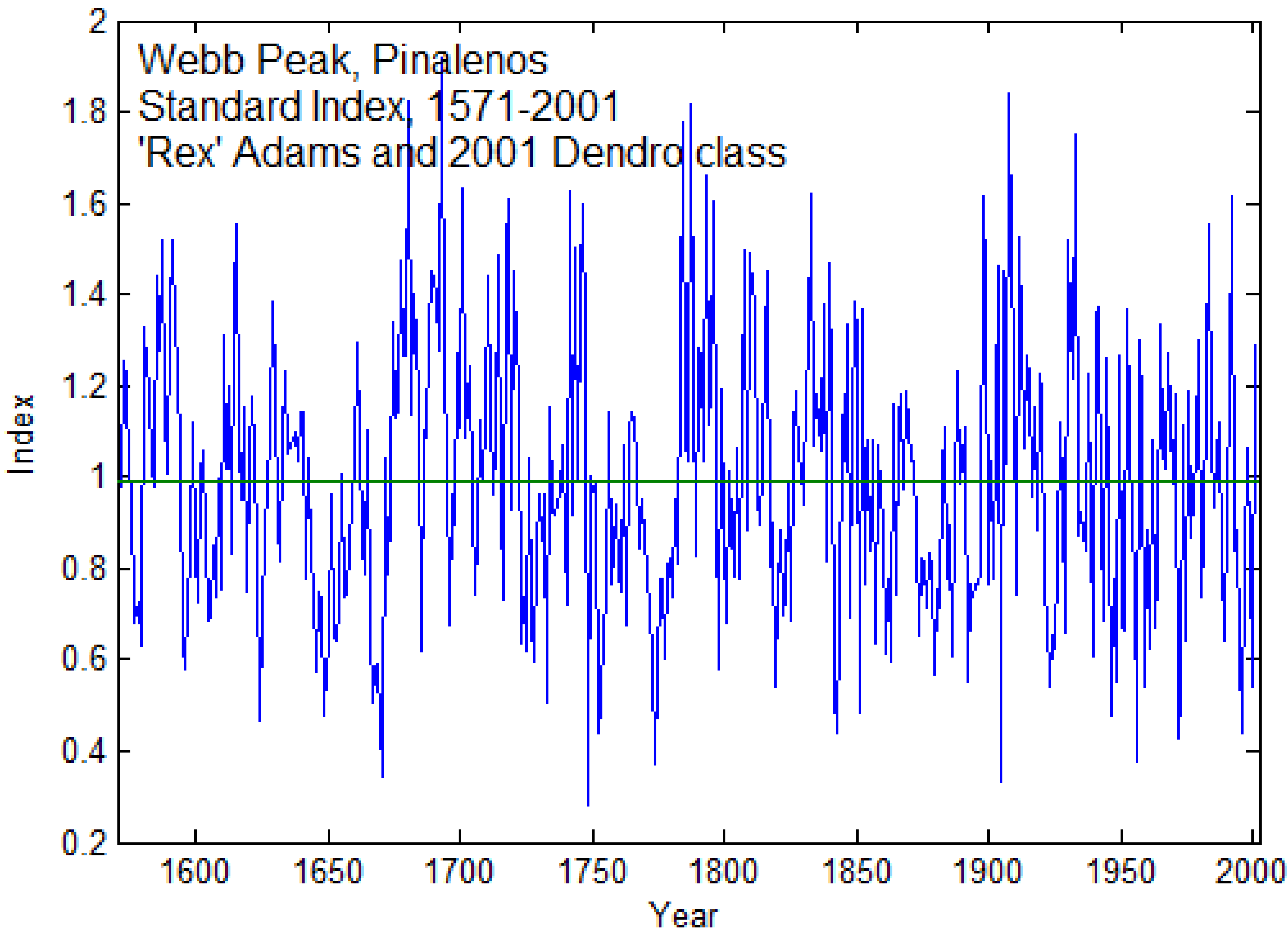


Outline of Smoothed Periodogram Method

1. Start with mean-subtracted time series, x_t
2. *Optionally pad (with zeros) and taper x_t
3. Compute discrete Fourier transform (DFT) of x_t
4. Compute raw periodogram from DFT
5. Smooth the raw periodogram

* next lecture

Time Plot of a Tree-Ring Index



Discrete Fourier Transform

“standard” frequencies

Recall that a time series can be expressed as the sum of periodic components at the Fourier frequencies

$$x_t = A(0) + \left\{ 2 \sum_{0<j<n/2} \left[A(f_j)\cos 2\pi f_j t + B(f_j)\sin 2\pi f_j t \right] \right\} + \left\{ A(f_{n/2})\cos 2\pi f_{n/2} t \right\}, t = 0, \ldots, (n-1)$$

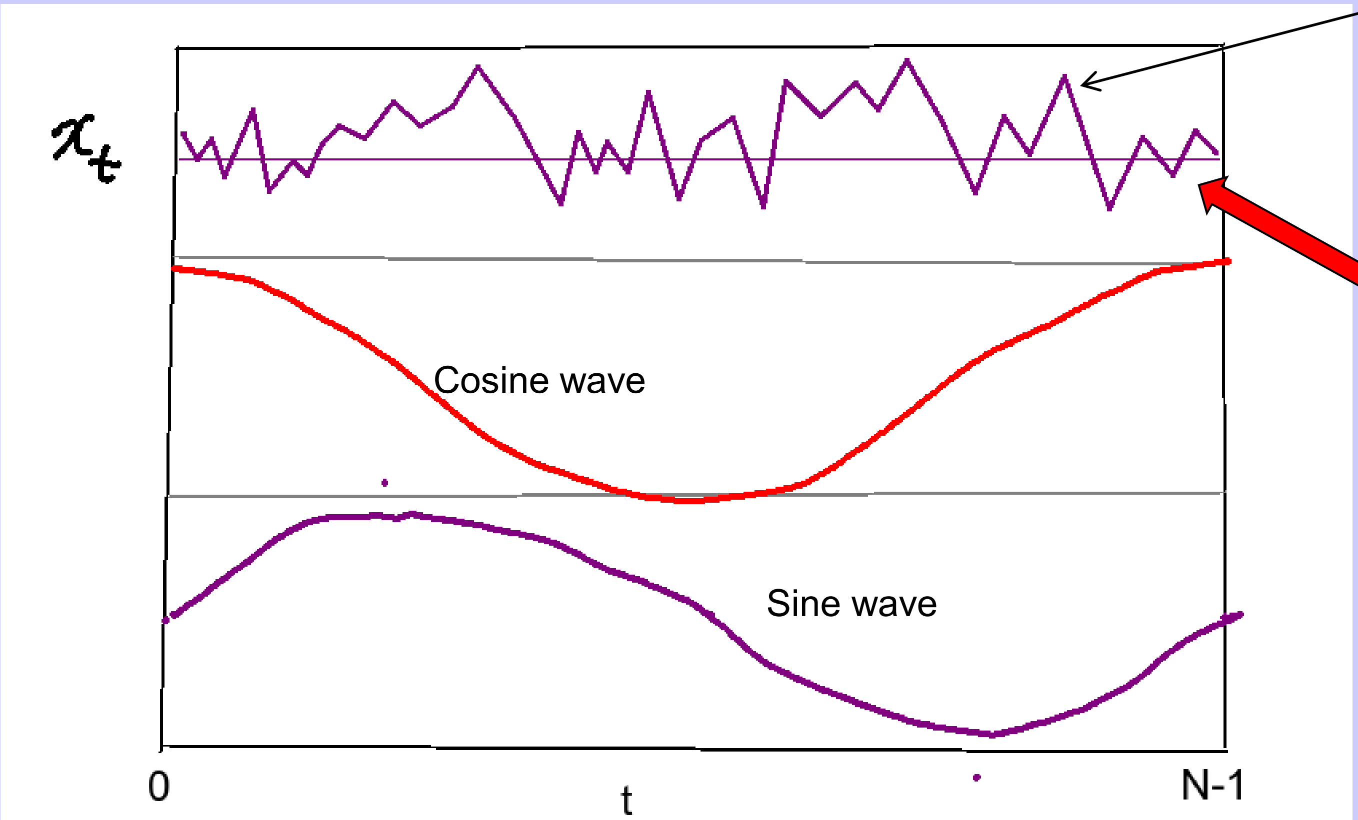
The coefficients in the above equation are computed as sums of products of the time series values and sines or cosines at the Fourier frequencies

Discrete Fourier transform
(DFT)

$$\begin{cases} A(f) = \dfrac{2}{n}\displaystyle\sum_{t=0}^{n-1} x_t \cos 2\pi f t \\ B(f) = \dfrac{2}{n}\displaystyle\sum_{t=0}^{n-1} x_t \sin 2\pi f t \end{cases}$$

Visualizing the DFT

Consider just the lowest Fourier frequency



time series

$x_t, t = 0, \ldots, (N-1)$

Size of $A(f_1)$ and $B(f_1)$ depends on strength of wave in the time series

Phase of any such wave in x_t is adapted to by relative importance of sine and cosine terms

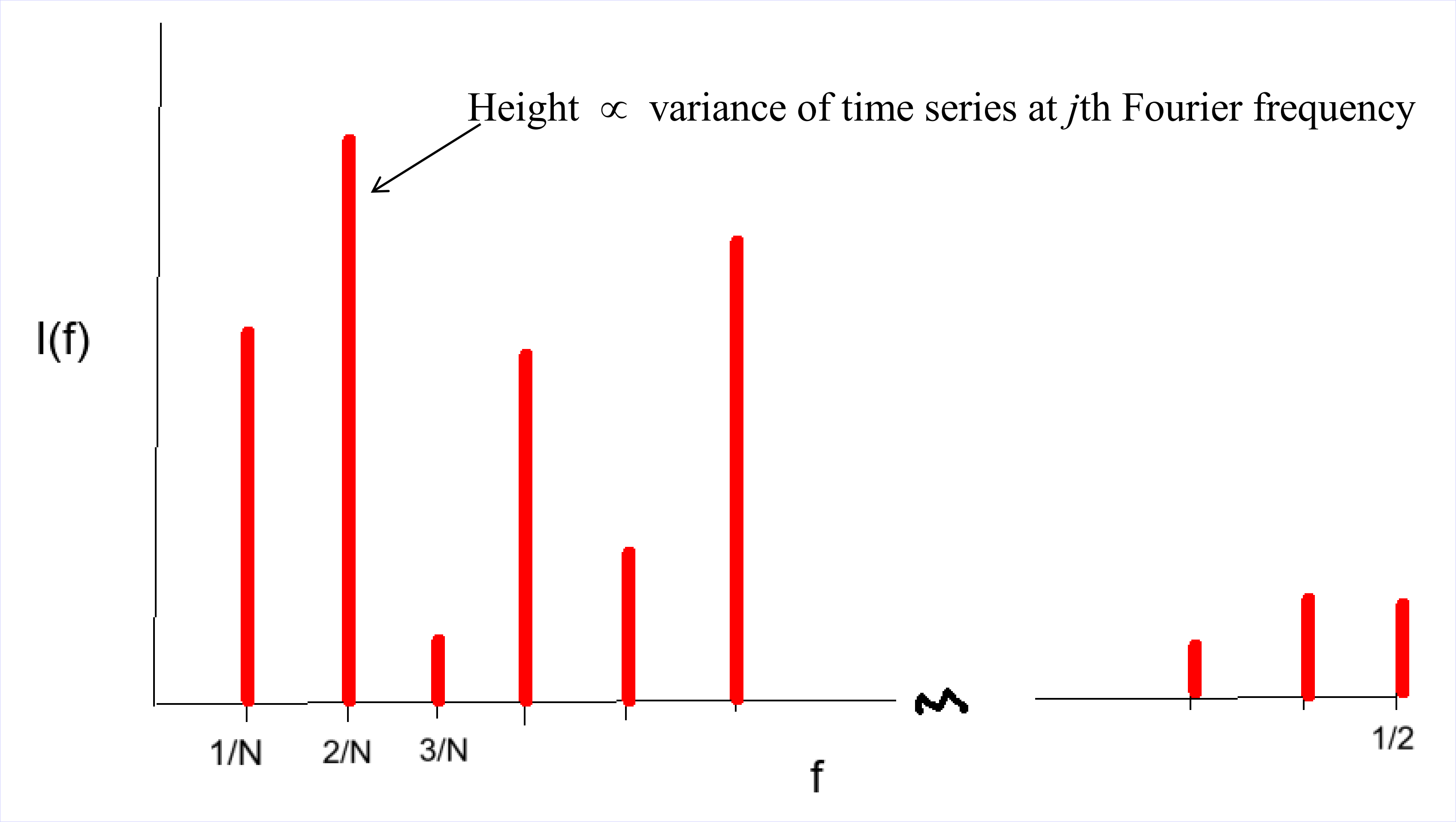
Periodogram

The periodogram ordinates are proportional to sum of squared amplitudes of the sine and cosine periodic components

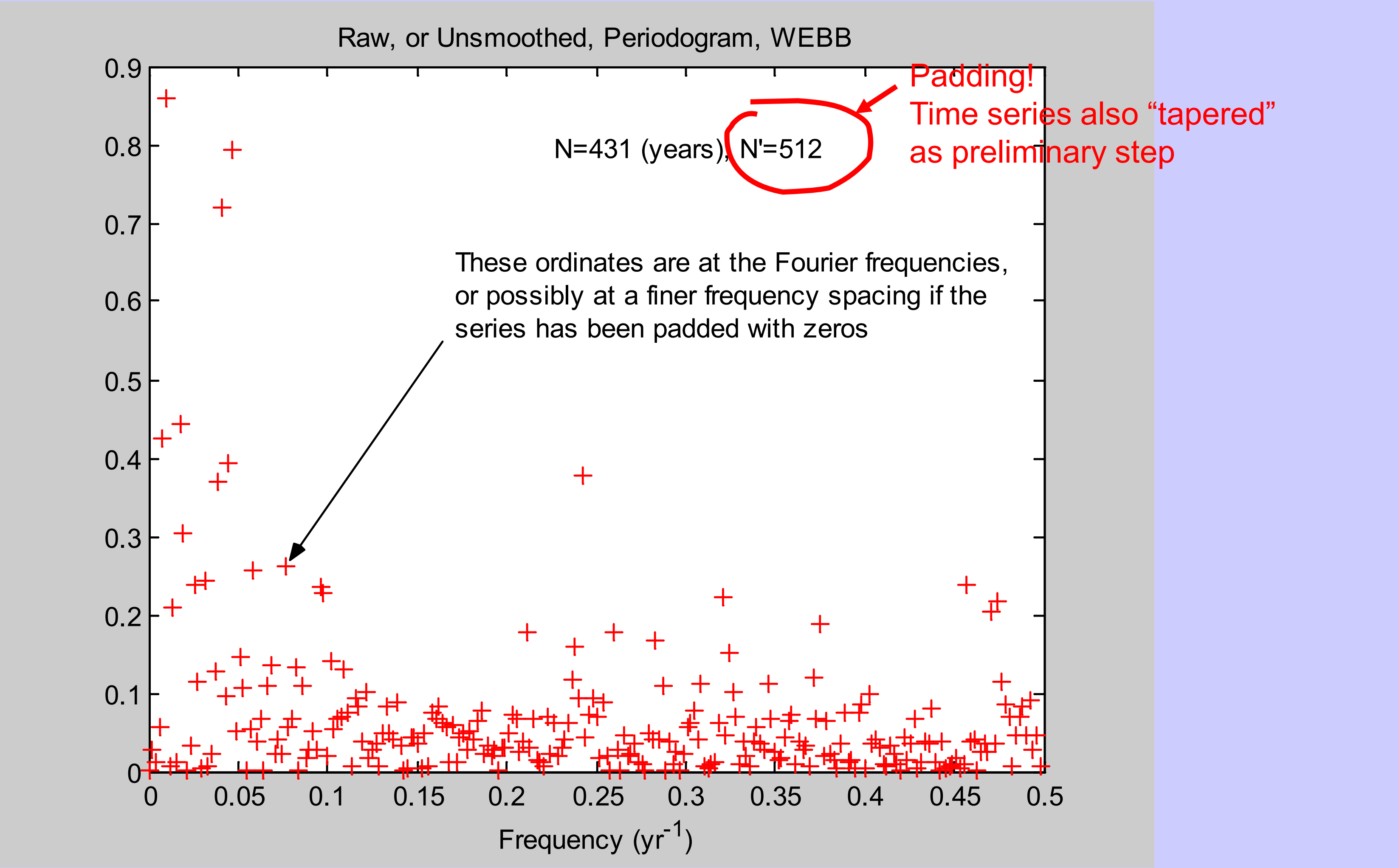
$$I\left(f_i\right) \propto \left(A_{f_i}^2 + B_{f_i}^2\right)$$

- A periodogram ordinate is proportional to the variance of the time series at the corresponding Fourier frequency
- The "raw" periodogram has high resolution: estimates at each Fourier frequency
- The raw periodogram typically has high variance, fluctuates wildly from ordinate to ordinate, and is considered an unstable estimator of the spectrum of the process
- The periodogram is smoothed to get a stable spectral estimate

Sketch of Periodogram



Raw Periodogram, Webb Peak Tree-Ring Index

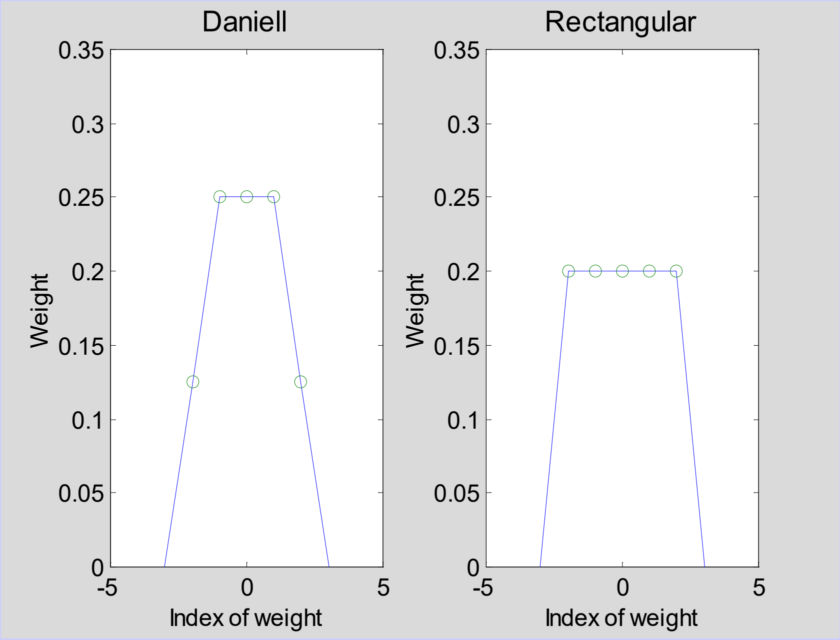


Smoothing the Periodogram

- How: apply spans of Daniell filters
- Objective: stable estimate of the spectrum
- Procedure: vary spans of the filters; observe effects
- Effect of increasing spans (length of filters)
    1. Smoother spectrum
    2. Decreased variance of spectral estimate
    3. Decreased resolution
- Increasing span has opposite effect of increasing lag window (M) in Blackman-Tukey spectral estimation

Basic Daniell filter

- Odd number of weights, symmetric, sum to 1
- End weight ½ size of other weights
- Used to smooth periodogram into spectral estimates
- Compare 5-weight Daniell and rectangular filter



- Both have 5 weights = “span”
- All weights positive
- Weights are equal for rectangular
- Two end weights are half as large as the other weights for Daniell

Convoluting simple Daniell filters

- In practice, the spectrum can be estimated by smoothing the raw periodogram with a “resultant” Daniel filter
- The resultant filter is produced by the mathematical operation of convolution of simple Daniell filters
- Convoluting two Daniell filters of spans m_1 and m_2 gives a resultant more bell-shaped filter with span ($m_1 + m_2$ -1)
- The weights of the resultant filter are symmetric, positive, and sum to 1
- Notation $\{m_1, m_2\}$ indicates result filter made by convolution of basic Daniell filters with spans m_1 *and* m_2

Convolution

"convolution"

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

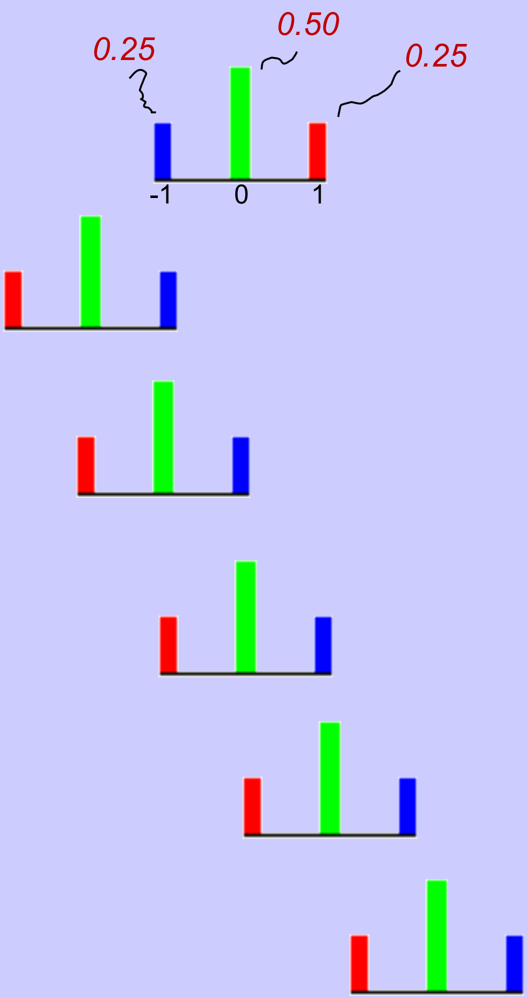
convolution of input signal with an impulse response...

can do similar operation with two Daniell filters

Convolution – graphical view

- Start with the two Daniell flter spans
- Reverse direction of one and slide along the another one weight at a time
- At each step, compute the a new, resultant, filter weight as the sum of the products of the two filters over their overlaps
- Example for convolution of two 3-weight Daniell filters to produce a 5-weight resultant filter …

“Resultant” Daniell Filter



Weights

0.0625

0.25

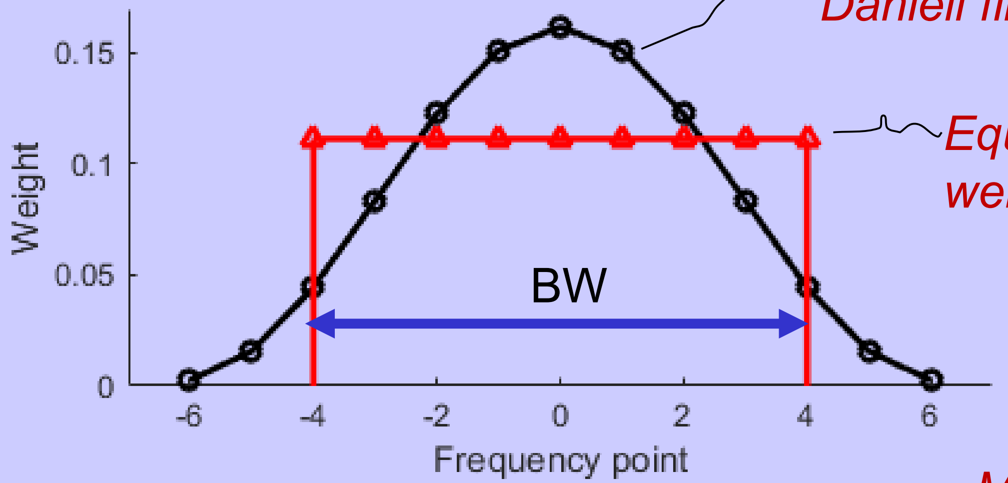
0.375

0.25

0.0625

Bandwidth of a spectral filter

BW = Bandwidth = width of “equivalent” rectangular filter



Daniell filter, weights w_i

Equivalent rectangular filter, weights $\boldsymbol{\omega}_i$

Might be at adjacent standard, or Fourier frequencies

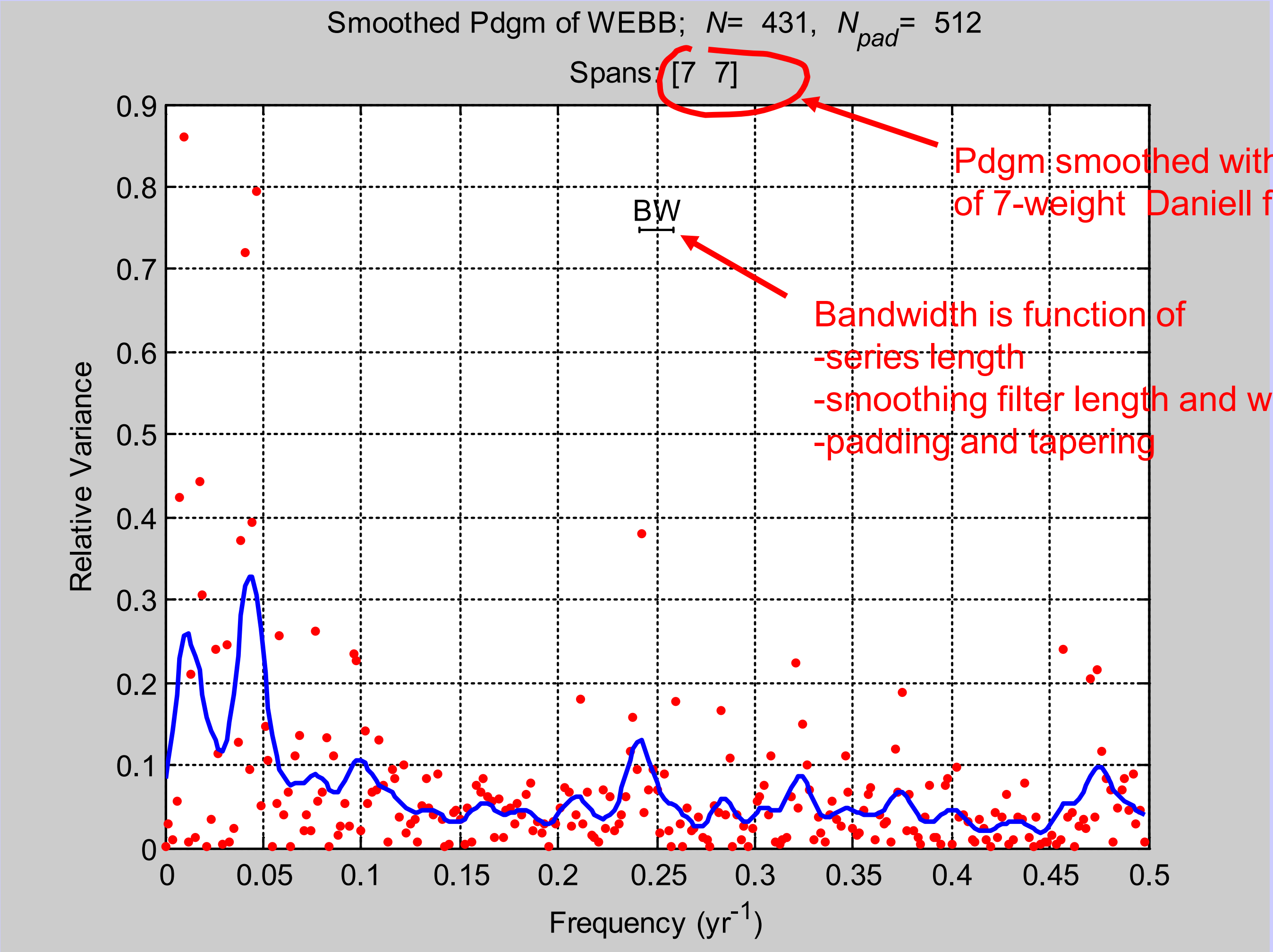
“Equivalent” -→ sum of squares of squares of weights ~equal

$$\sum_{\text{all } i} w_i^2 = \sum_{\text{all } i} \boldsymbol{\omega}_i^2$$

Bandwidth control

- Through setting spans of Daniell filter in the spectal analysis
- Determines how many adjacent standard frequencies the spectrum is averaged over
- Example for spectrum of a tree-ring time series …

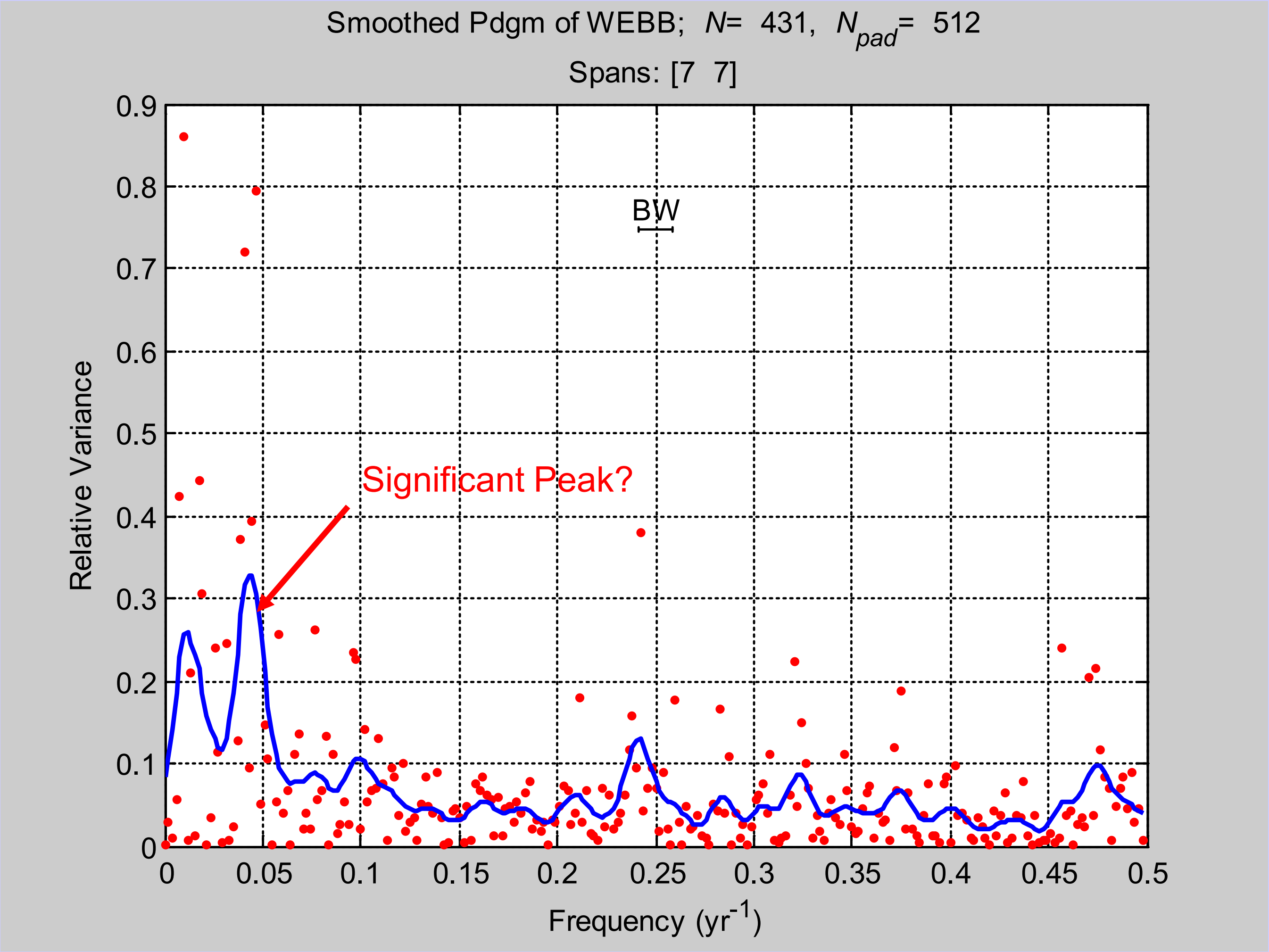
Example of Smoothed Periodogram: Webb Peak tree-ring index, 1571-2000



Pdgm smoothed with two spans of 7-weight Daniell filters

Bandwidth is function of
-series length
-smoothing filter length and weights
-padding and tapering

Significance of Spectral Peaks
Testing for Periodicity



Distribution of Spectral Estimates (see p 184 in Bloomfield (2000))

- Periodogram estimates are independent and approximately exponentially distributed
- Smoothed periodogram spectral estimate can be considered a sum of periodogram ordinates
- Smoothed periodogram spectral estimate is approximately chi-squared distributed

The 95% CI for the spectral estimate

Degrees of freedom for appropriate chi squared distribution is inversely proportional to sum of squares of Daniell filter weights (see eqns 21, 22 in notes)

Spectral estimate

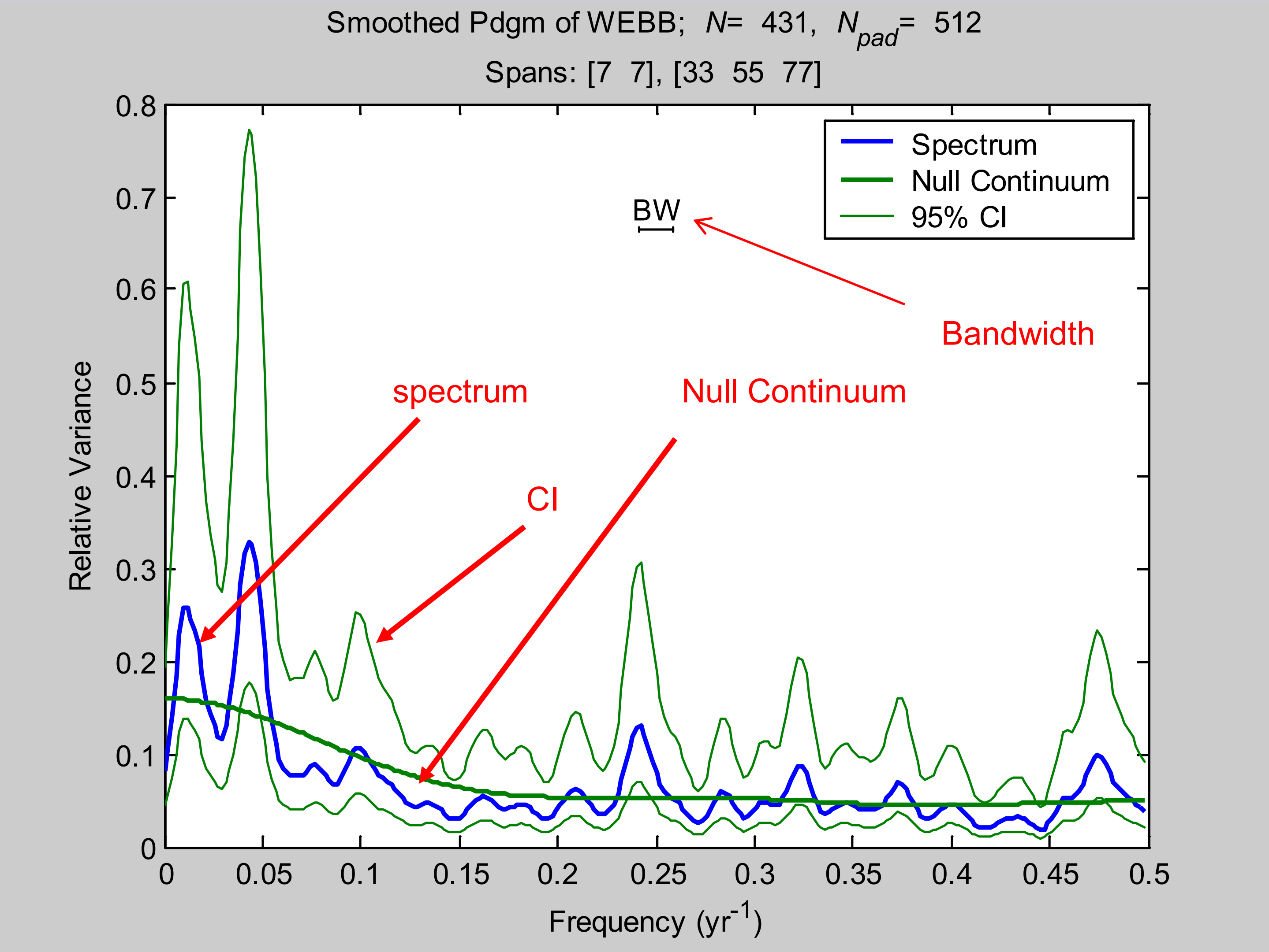
$$\frac{\nu\hat{s}(f)}{\chi^2_\nu(0.975)} \le s(f) \le \frac{\nu\hat{s}(f)}{\chi^2_\nu(0.025)}$$

Eqn 16 in notes

0.975 prob. point of cdf of chi squared distribution with v degrees of freedom

(Unknown) spectrum of process

Example of spectrum with CI: Webb Peak Tree-Ring Index, 1571-2000



Considerations in Evaluating Peaks

1. Null continuum: peak significantly different from what?
2. Multiple comparisons: caveats on fishing expeditions