Tues, 3-19-19

Filtering

* Lightning talk
* Feedback on A7

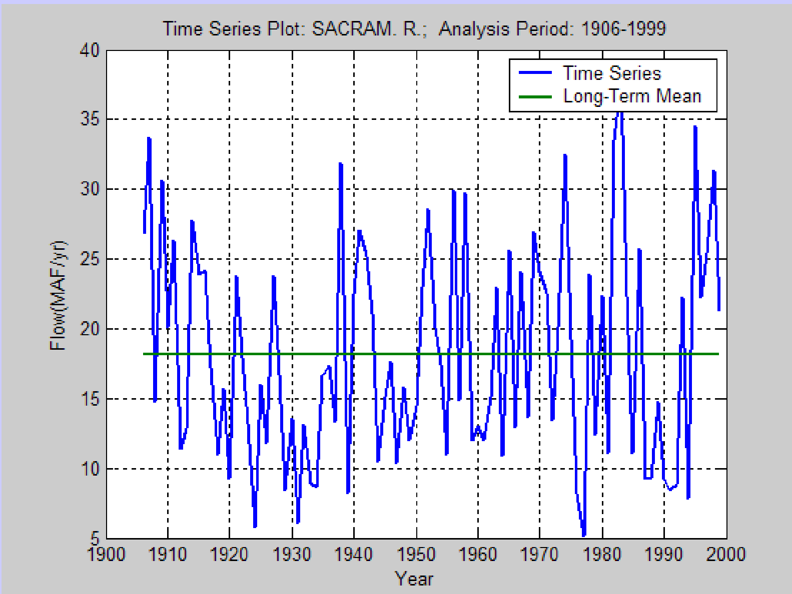
1. **Lowpass filtering**
2. **Filter design**
3. **Frequency response of filter**

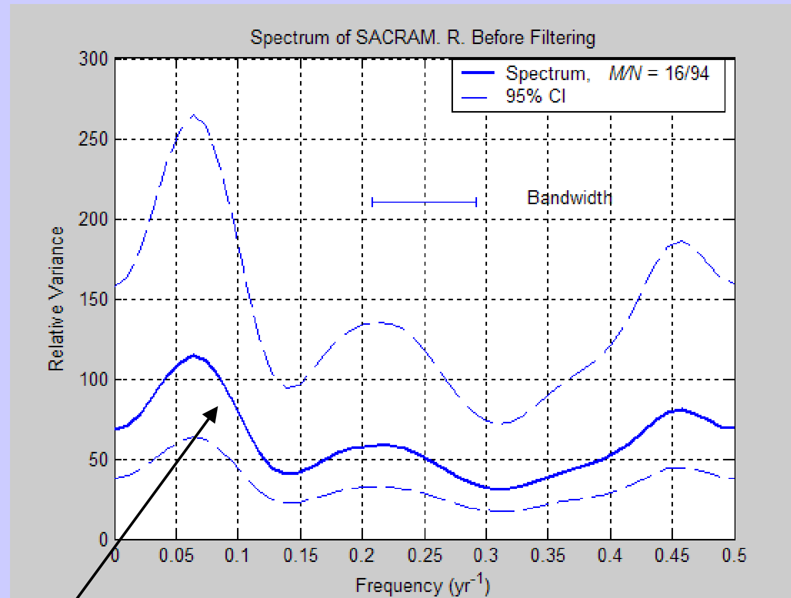
Read notes_8.pdf

A7 Feedback

1. Download A7x.pdf from D2L
2. Automatic points, for running assignment and having uploaded by due time, is already marked in parentheses at top of first page
3. Each assignment has maximum possible 10 points; if you make no deductions, score is 10/10
4. A7x is color coded for points; purple=1; yellow=0.5; blue=0.5
5. Open your copy of the same assignment pdf you uploaded
6. In Acrobat Reader, using “Add text box,” mark in right margin for deductions only, with deduction and segment reference : (eg., -0.5 A); round to tenths in deductions (e.g., no -0.25)
7. At top of your pdf, mark grade like this : 9.5/10
8. If necessary, put any comments at top near the grade
9. Upload your self-graded pdf to folder A7_graded in D2L

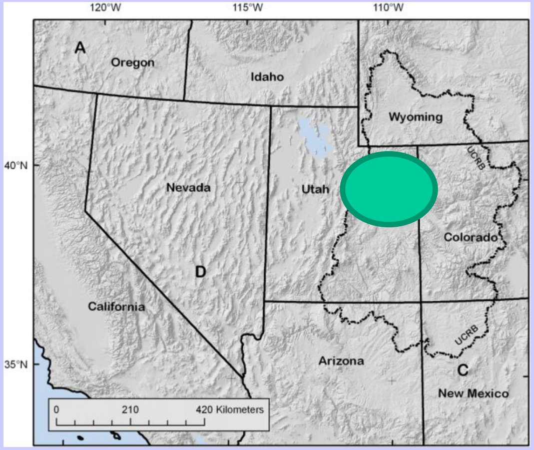
Goal of filtering: emphasize some frequencies and suppress others





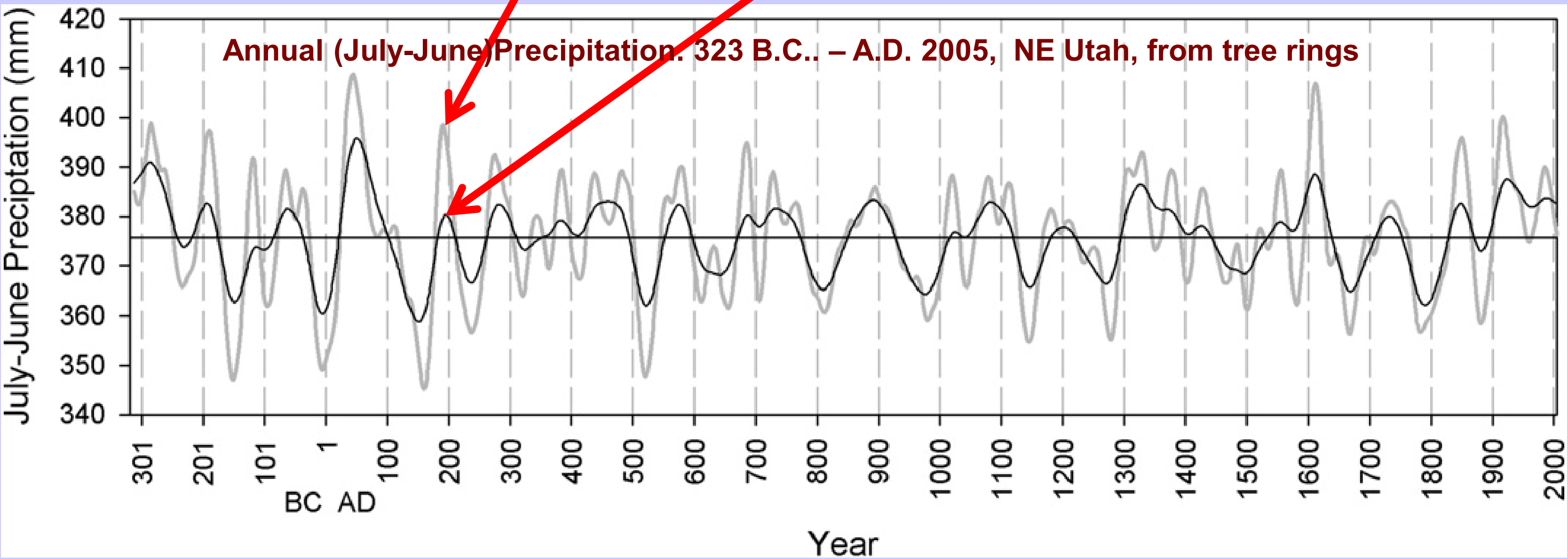
Spectrum shows series has much variance
At low frequencies. May want to study that while reducing the confusing influence of interannual variations

Cubic smoothing spline can be uses as a filter



50-yr spline

100-yr spline



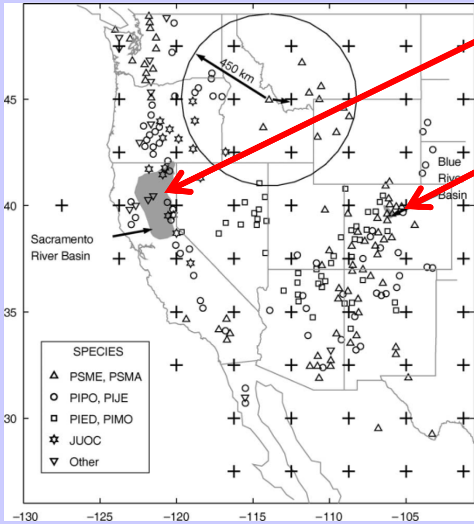
Knight et al. (2010), Quaternary Research 73, 107-117

Symmetrical Lowpass Filters

- Emphasize low-frequency variations
- Also called “smoothing” filter – smooths out high-frequency variations

1. **Low-pass**
2. **High-pass**
3. **Band-pass**

Example: Smoothed tree-ring reconstructions of river flow

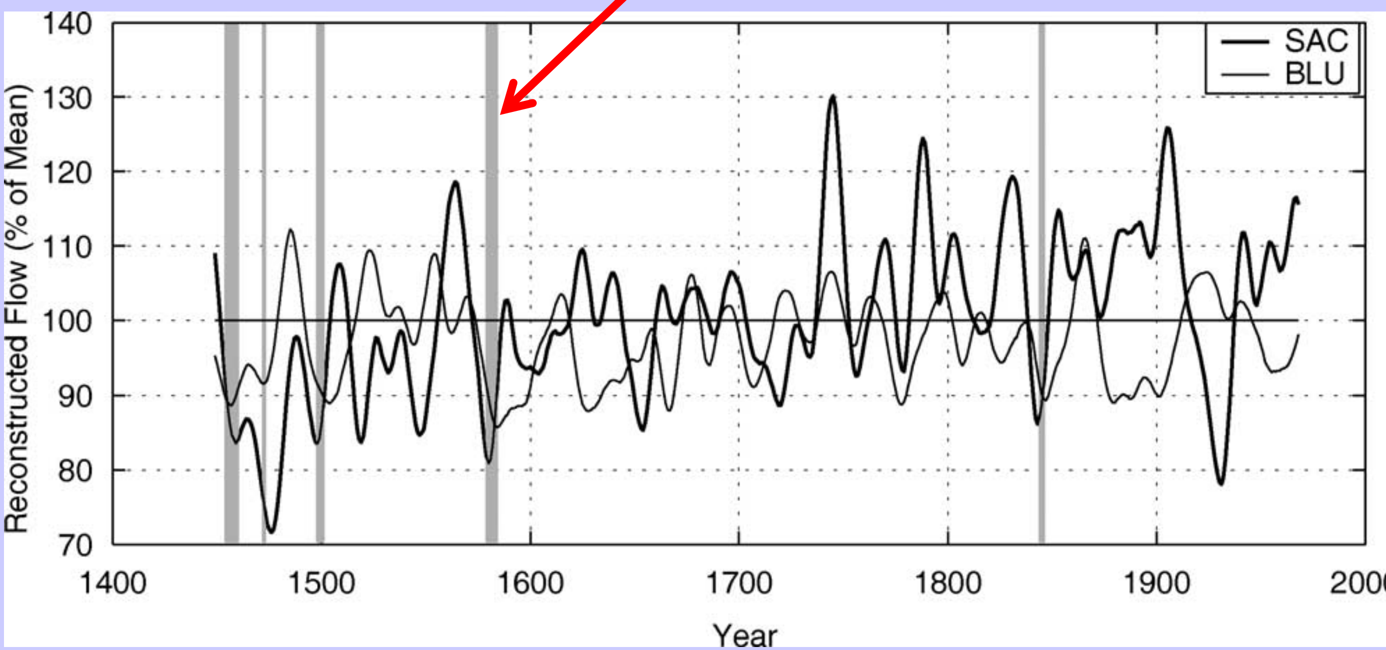


Sacramento River Basin

Blue River Basin

- Gaussian filter
- 50% frequency response at wavelength 20 yr

Both smoothed series below 0.2 quantile



Meko and Woodhouse (2005), J. of Hydrology 308: 196-213

Digital filter—mathematical operation

Smoothed series

Weights

Original series

$$s_t = \sum_{i=-n}^{n} w_i x_{t+i}$$

$$\sum_{i=-n}^{n} w_i = 1$$

- Weights sum to 1
- Symmetrical filters most useful
- Filter usually <u>centered</u>

Digital Filter

Year	Filter	Time Series	Filtered Values
1		12	
2	.25 x	17	14.00
3	.50 x	10	14.75
4	.25 x	22	17.25
5		15	15.75
6		11	13.75
7		18	18.50
8		27	21.50
9		14	

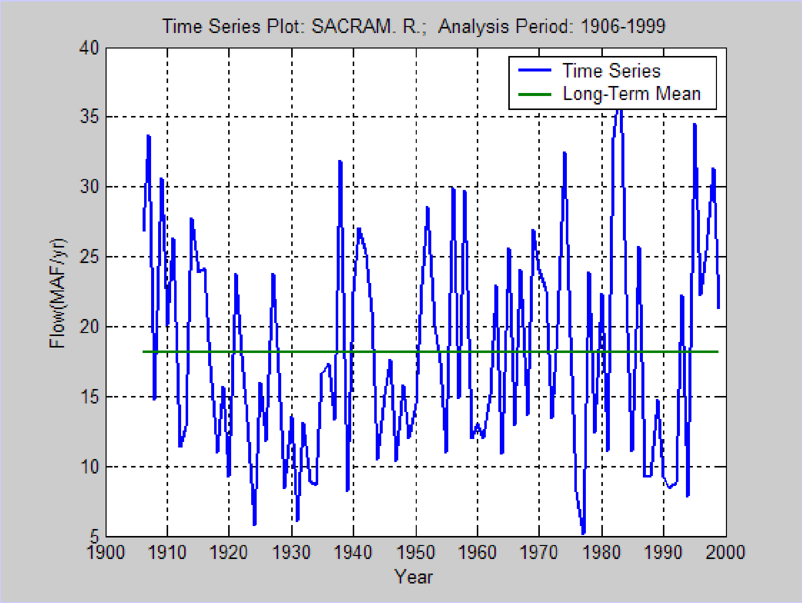
Filter Span

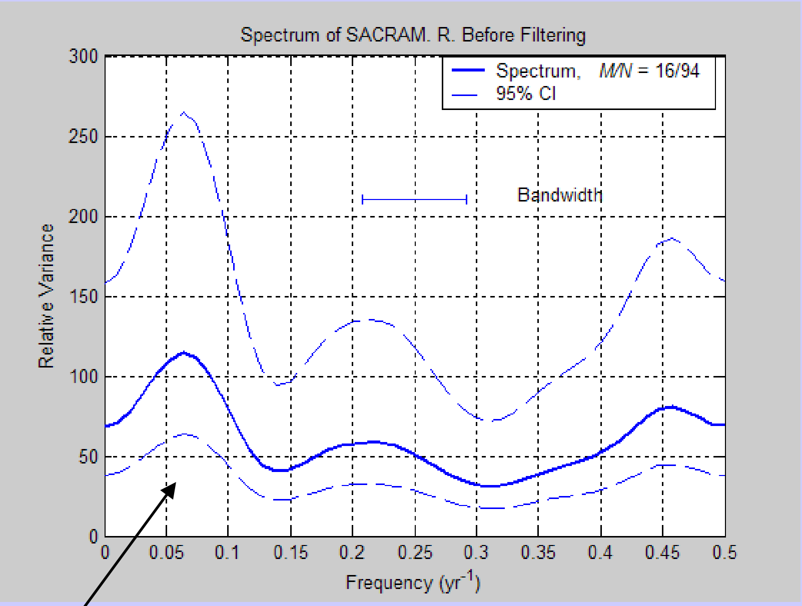
$$s_t = \sum_{i=-n}^{n} w_i x_{t+i}$$

- We will be considering symmetric filters with an odd number of weights
- The "span" is defined as max(i)-min(i)+1, where i is a relative time index to the center of the filter; a symmetrical filter weighting 5 observations, from i=-2 to i=2 therefore has span 5. The span equals the number of weights.
- Applying the filter centered on the original data points leads to a filtered series shorter than the original series. The total number of observations "lost" from both ends is one less than the span.
- Other things being equal, it is desirable for a filter to have a <u>short span</u>

Filter Design

What is the frequency-range of interest?
At what frequencies is variance "large"?





Spectrum shows series has much variance
At low frequencies. May want to study that while
reducing the confusing influence of interannual
variations

Design Considerations

1. Impulse response
2. Frequency response
3. End effects

Impulse Response Function (IRF) of Digital Filter

(response to a “unit pulse” of input)

$$s_t = \sum_{i=-n}^{n} w_i x_{t+i}$$

$x_t =$ hypothetical input series, zero at all times except one, at which

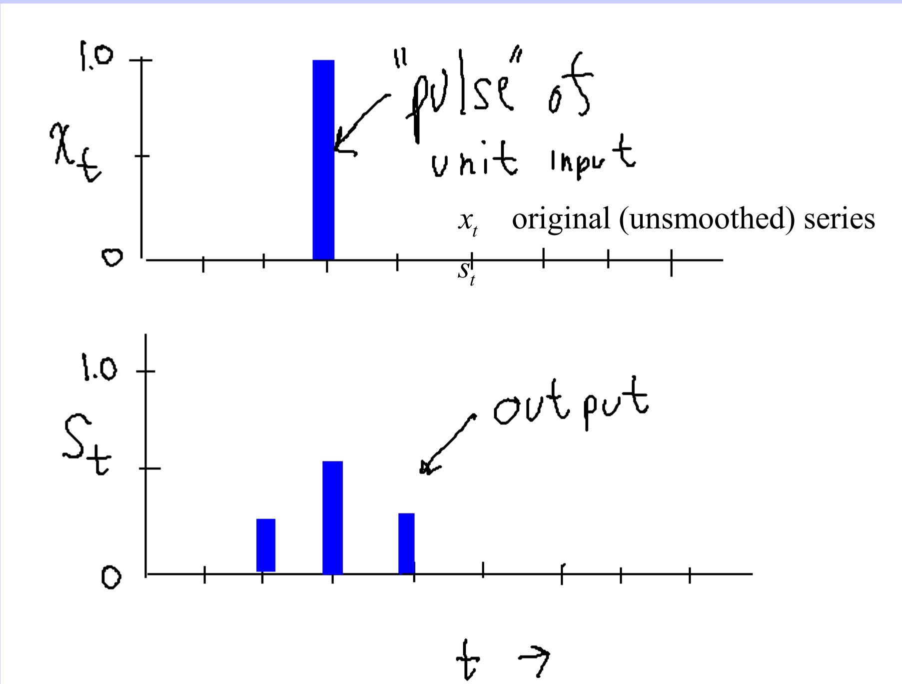
$x_t = 1$ (unit pulse)

$w_i =$ filter weights (symmetrical, sum to 1)

Output, s_t, is identically equal to the filter weights

Example: finite impulse response (FIR) filter

Consider simple 3-weight filter



x_t original (unsmoothed series)

s_t smoothed series

$w_i = \{0.25, 0.50, 0.25\}$ weights

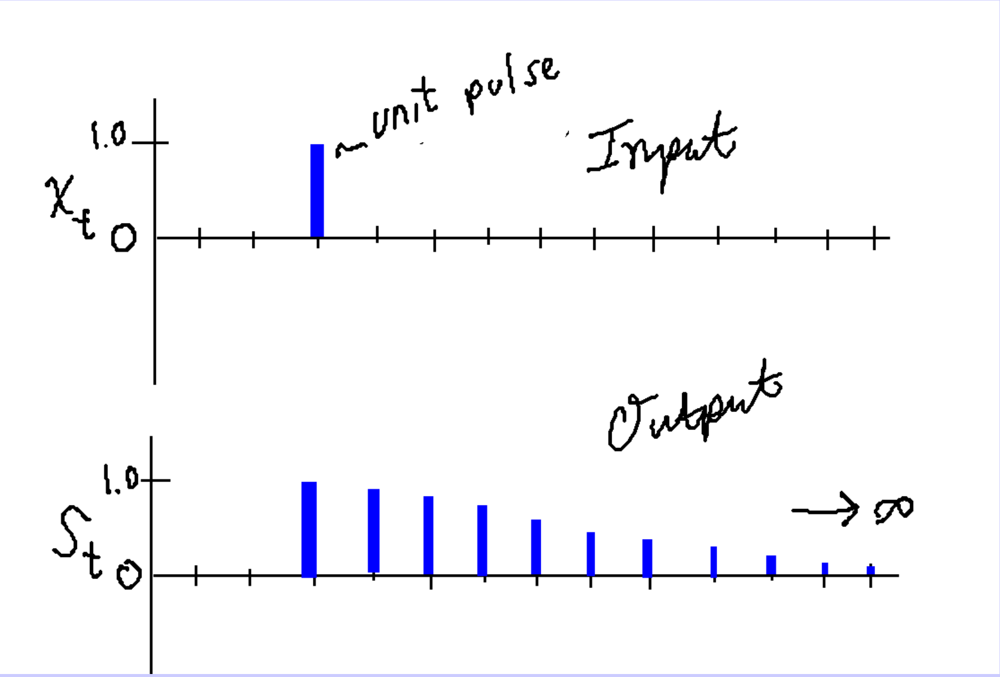
$s_t = \sum_{i=-1}^{1} x_{t+1}$ filtering equation

Response finite: unit pulse of input at some specific time results in output response limited to a finite number of observations

- Symmetric: odd number of weights, symmetric around central weight
- Zero phase: does not shift location of peaks or troughs in the original series
- Acausal: first "output" can occur BEFORE time of unit pulse of input

Example: infinite impulse response (IIR) filter

Consider simple recursive filter



x_t original (unsmoothed series)

s_t smoothed series

$s_t = 0.9 s_{t-1} + x_t$ filtering equation

Response infinite: unit pulse of input at some specific time results in output response that never ends (in practice could become very small)

For this example, filter is:

- Asymmetric: weights do not regularly decrease in both directions from a central weight
- Not zero phase: could shift the location of peaks or troughts in the original series
- Causal: first "output" cannot occur BEFORE time of unit pulse of input

Data extension and end effects

$x_t, t = 1, N$ time series to be filtered

Symmetric filter of length m

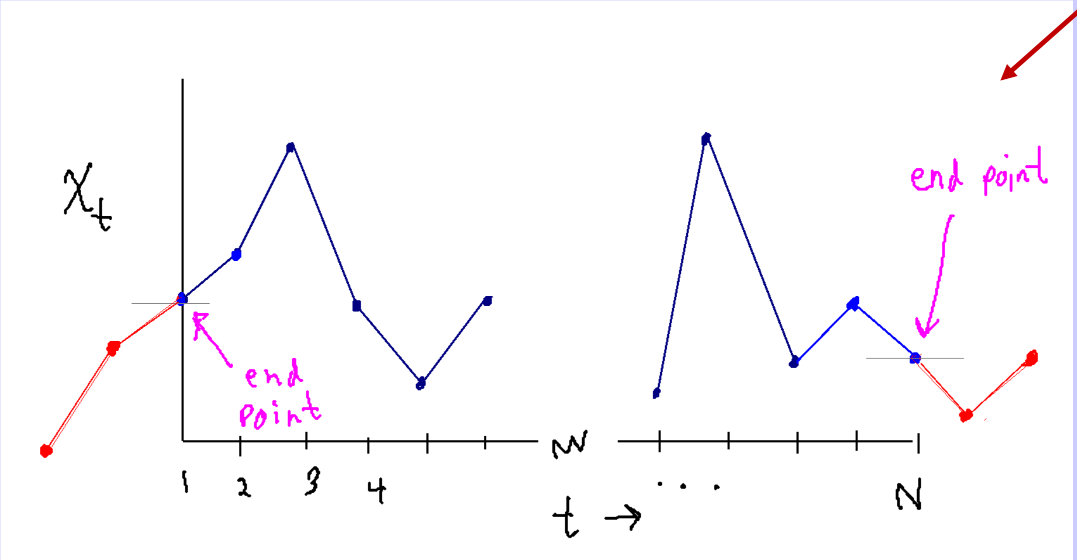
Lose $\dfrac{(m-1)}{2}$ observations in filtering,

unless assign fake data beyond ends of

x_t before filtering

options

1. Accept the loss of data
2. Pad series with some constant (e.g., the mean)
3. Repeat the end-point data beyond the end points
4. Reflect data across end points



One advantage would be to avoid distortion in time series with trend in mean (in contrast to simply extending with the mean or median)

Frequency response of a filter

- Describes the hypothetical effect of the filter on sinusoidal inputs of various frequency: can affect amplitude and phase
- For a symmetical filter, phase change is zero, and amplitude is multiplied by factor equal to amplitude of the frequency response function
- Spectrum is affected as follows:

$$S_y(f) = G^2(f) S_x(f), \quad \text{where}$$

$S_x(f)$ is the spectrum of the original series at frequency f

$S_y(f)$ is the spectrum of the smoothed series at frequency f

$G^2(f)$ is the squared amplitude of the frequency response function at frequency f

Equation for Frequency Response of Symmetrical Filter

Frequency

Time interval

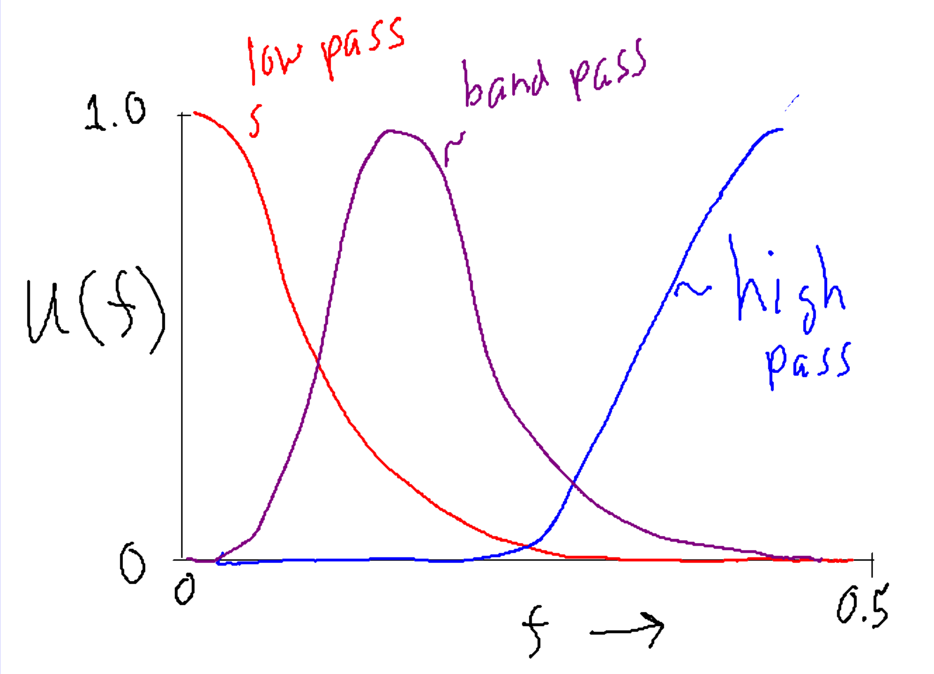
$$u(f) = w_0 + 2\sum_{k=1}^{n} w_k \cos(2\pi f k \Delta t)$$

Amplitude of response

Central Weight

k-th weight, numbered away from central weight

Frequencies "passed"



$x_t, t = 1, N$ time series

s_t smoothed series

$U(f)$ frequency response

Frequency response summarizes how well variations at different frequencies in the original series are retained in the filtered series

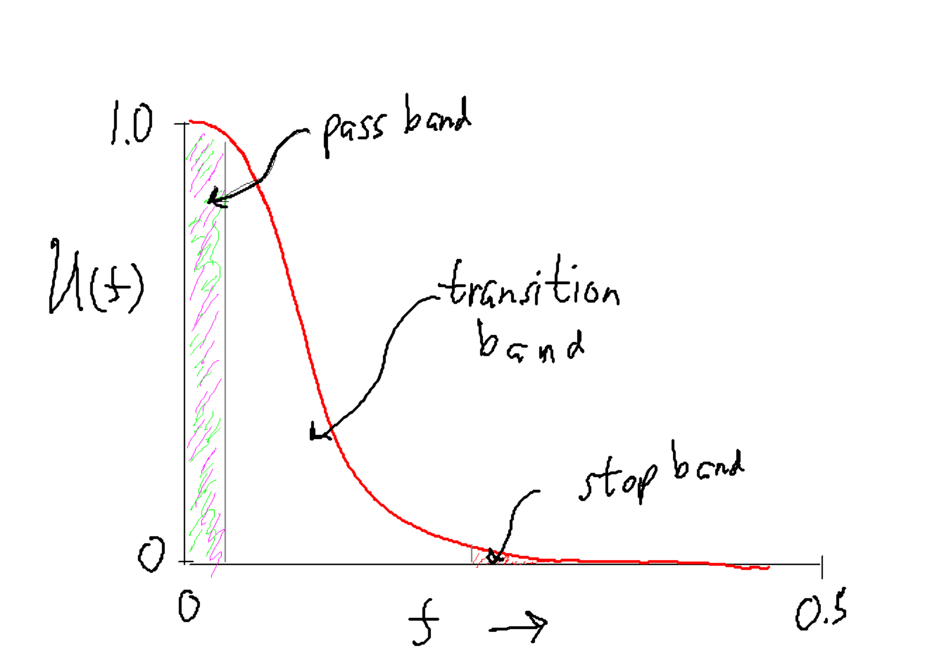
Detrending is a form of high-pass filtering

If s_t is a lowpass (smoothed) version of x_t,

$r_t = x_t - s_t$

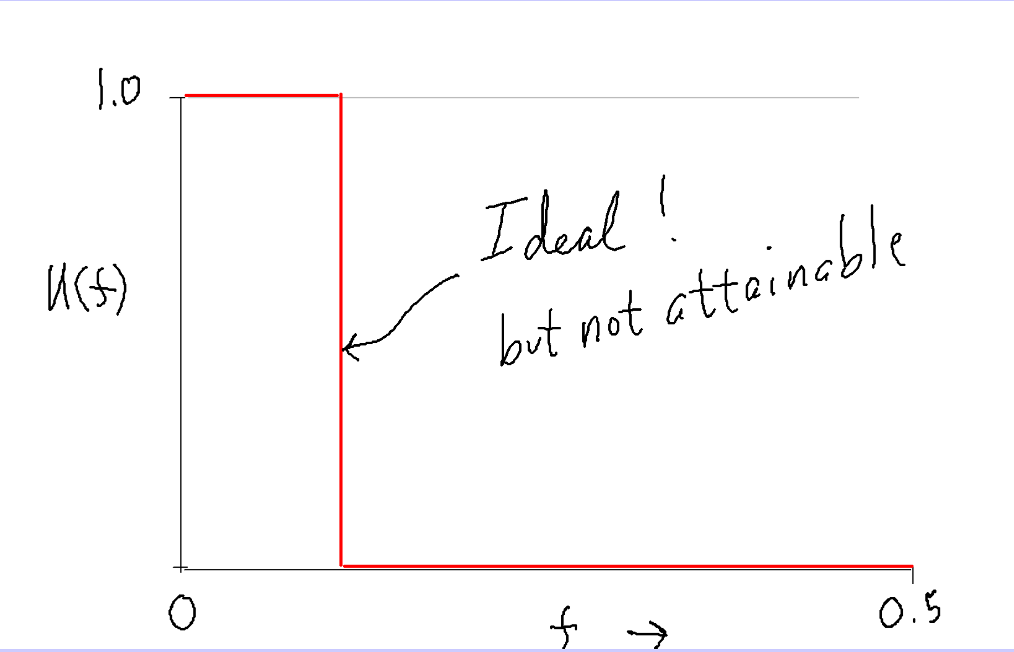
is the highpass component of x_t

Frequency response relationship to variance



- Frequency response is the ratio of amplitudes of waves before and after filtering
- Frequency response function describes the effect of smoothing on variance tracked as a function of frequency
- With a lowpass filter, variance tracked is variance retained
- Tracking is intermediate between strong and weak in the transition band

“Ideal” or “brick wall” filter



- Would perfectly pass all variations at frequencies lower than some cutoff frequency and totally eliminate variations at frequencies higher than the cutoff frequency
- Symmetic filters of finite length cannot achieve this

Effect of smoothing on the spectrum of the time series

If

x_t input (original series)

y_t output (filtered, or smoothed, series)

S_x spectrum of x_t

S_y spectrum of y_t

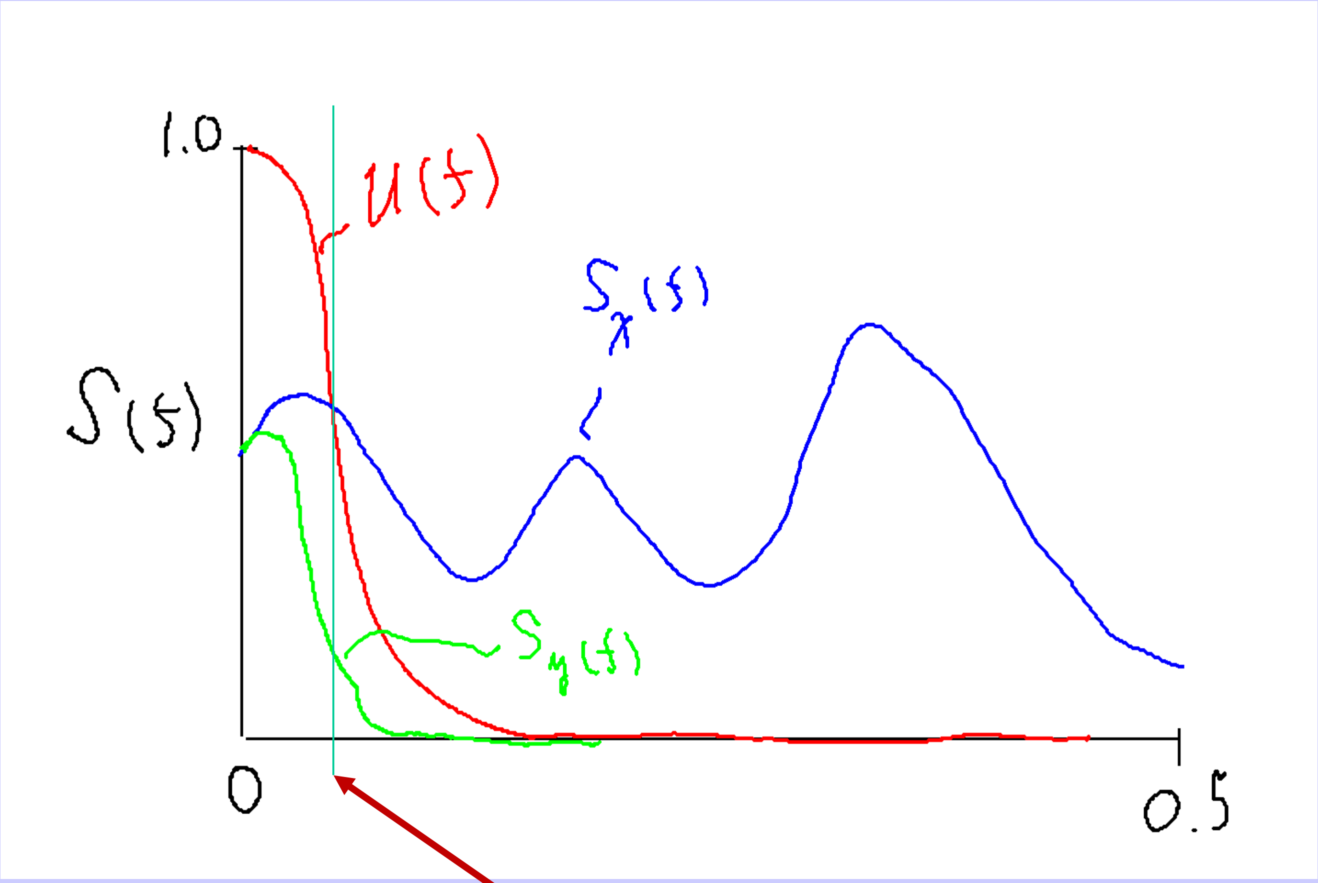
u(f) frequency response function of smoothing filter

then

$$S_y(f) = u^2(f) S_x(f)$$

The squared frequency response is equal to the ratio of the spectrum of the smoothed series to the spectrum of the original series

Example



Where frequency response drops to 0.5, spectrum of smoothed series is about ¼ the height of spectrum of the original series