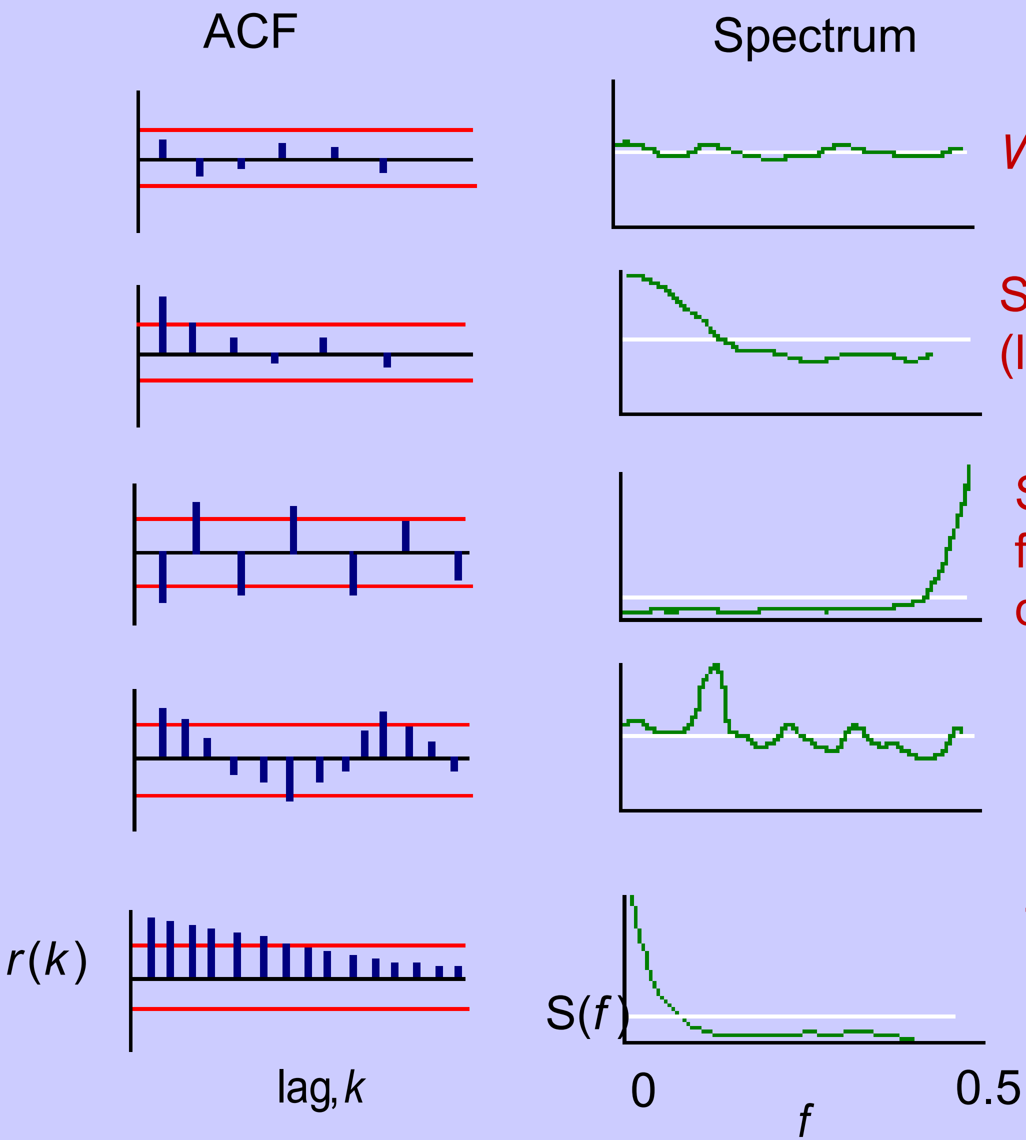
Thurs, 2-14-19

4. Spectrum (cont.)

1. Characteristic spectra
2. Estimating the spectrum
3. Blackman-Tukey estimation method
4. Sample runs of geosa4

– Assignment a4: due Tues, Feb 19

Characteristic Spectra



White noise

Short-term memory (low-frequency spectrum)

Series tending to go back cnd forth across mean (up one year, down the next)

Periodic series

Trend

Estimating the Spectrum

$f(\omega)$

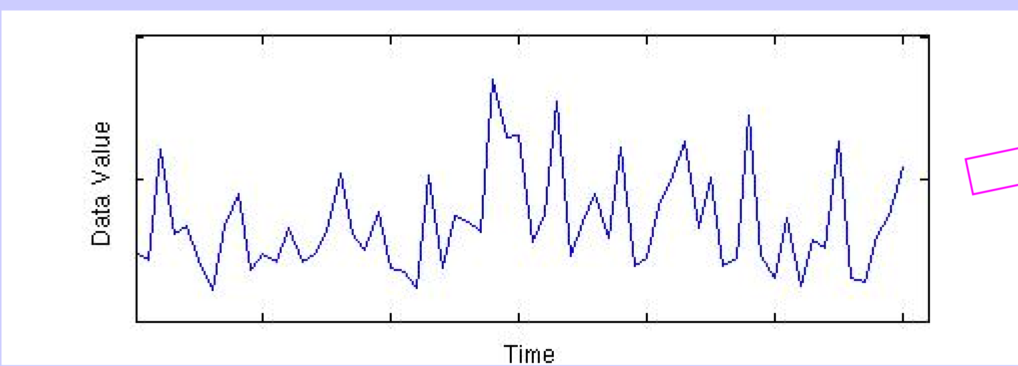
Theoretical Spectrum
(applies to population)

Estimated Spectrum

$\hat{f}(\omega)$

$c(k)$

Time Series



Sample Autocovariance

Notation

$$f(\omega)$$

Population spectral density, as function of angular frequency

$$\hat{f}(\omega)$$

Sample spectral density function (the estimate of the population spectral density that we get from the data

distinguish

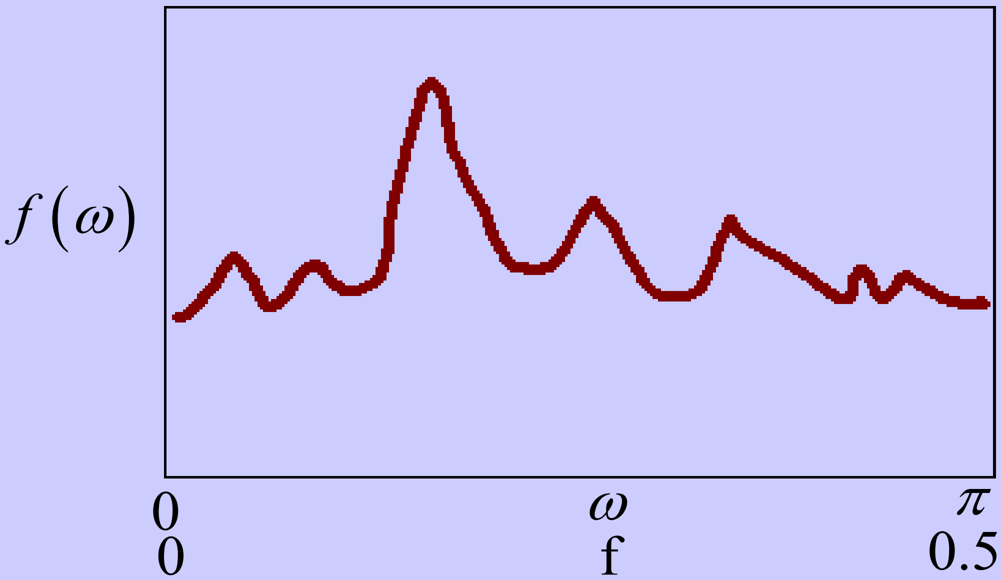
$$\mathrm{f} = \frac{1}{\lambda}$$

Frequency, as “cycles per time unit”, defined as the inverse of the wavelength

$$\omega = 2\pi \mathrm{f}$$

Conversion of frequency to angular frequency

Population spectrum



- Unknown
- Assumed to be "smooth" (i.e., not a line spectrum)
- Estimated from the data
- The estimate will have properties that vary depending on the data and estimation procedure:
    - "variance", or uncertainty
    - bias –tendency to be shifted vertically in a preferred direction from the true value
    - resolution – ability to identify peaks at finely spaced frequencies

Alternative ways to estimate the spectrum

- Smooth the amplitudes of the *N*/2 harmonics derived from a classical periodogram or harmonic analysis

- Fourier analysis of the autocovariance function*

We use this method in this week's assignment

Blackman-Tukey spectral estimation method

Relationship between spectrum and acvf

$$f(\omega) = \frac{1}{\pi}\left[\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos\omega k\right].$$

- Spectrum of a process is the Fourier transform of the autocovariance function of the process
- Note that the summation extends to lag infinity
- Spectral estimation by Blackman-Tukey method uses Fourier transform of a **TRUNCATED, SMOOTHED** sample acvf

Blackman-Tukey Spectrum

Weights

Sample acvf

$$\hat{f}(\omega) = \frac{1}{\pi}\left\{\lambda_0 c_0 + 2\sum_{k=1}^{M} \lambda_k c_k \cos \omega k\right\}$$

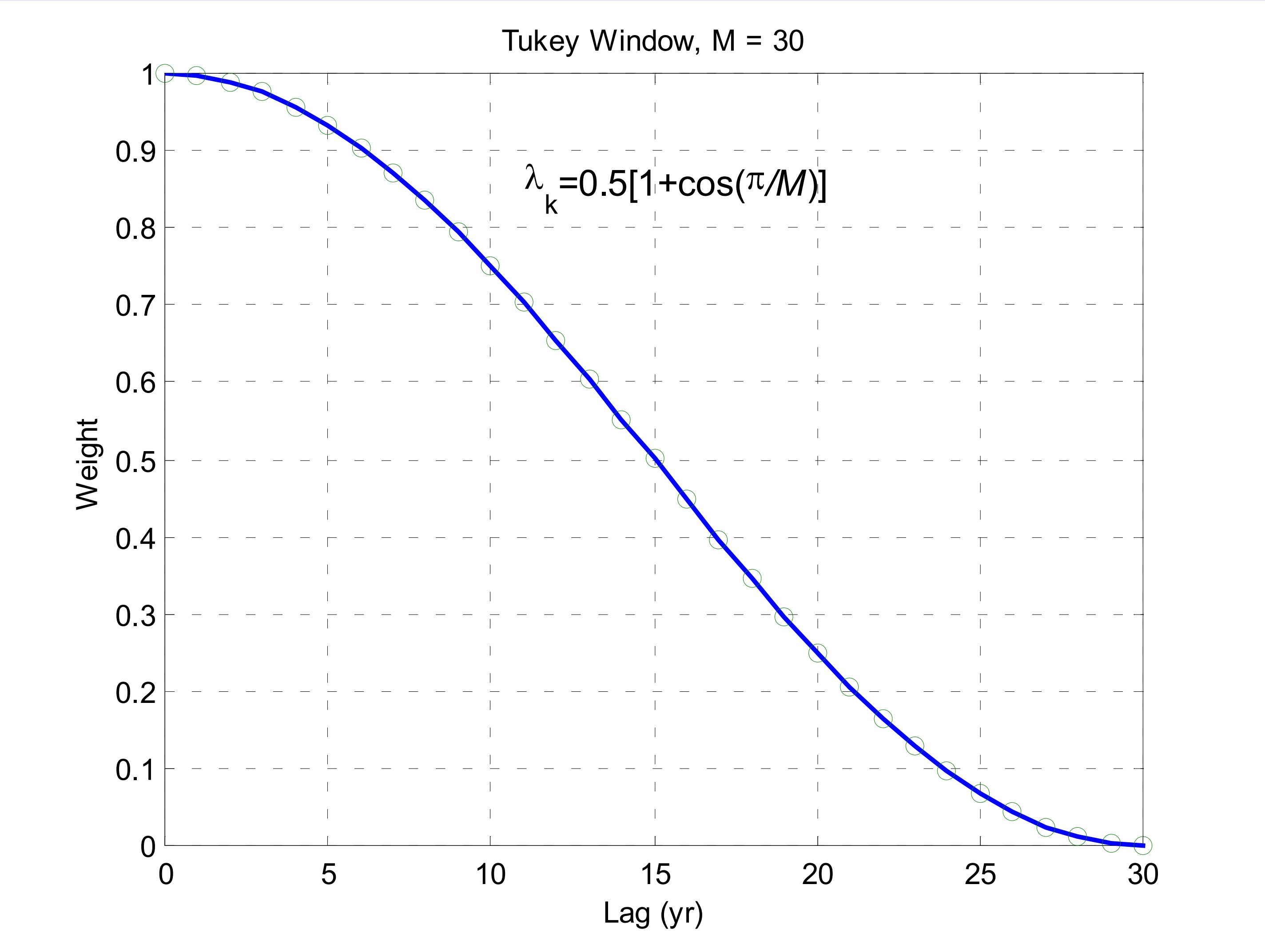
Lag window

$$\lambda_k = 1\Big/2\left(1 + \cos\frac{\pi k}{M}\right) \qquad k = 0, 1, \cdots, M$$

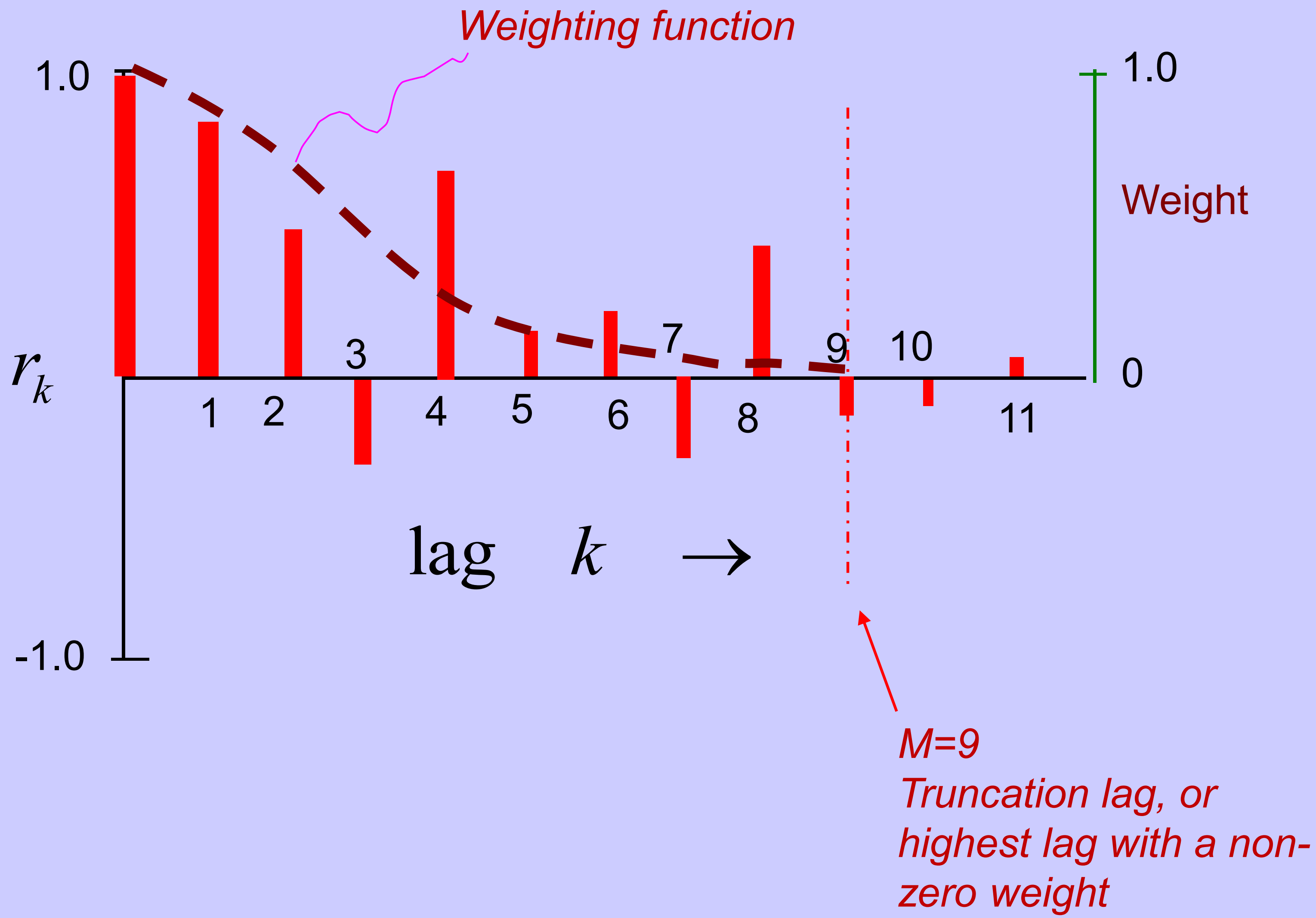
Truncation point

M

Plot of Tukey Window for M=30



Lag window


Weighting function
1.0
-1.0
r_k
1 2 3 4 5 6 7 8 9 10 11
lag k →
1.0
Weight
0
M=9
Truncation lag, or highest lag with a non-zero weight

Choice of *M* affects these spectral properties:

1. Bias
2. Variance
3. Resolution

Window closing approach

1. Start with a value of *M*

2. Compute and plot spectrum

3. Change the value of *M*

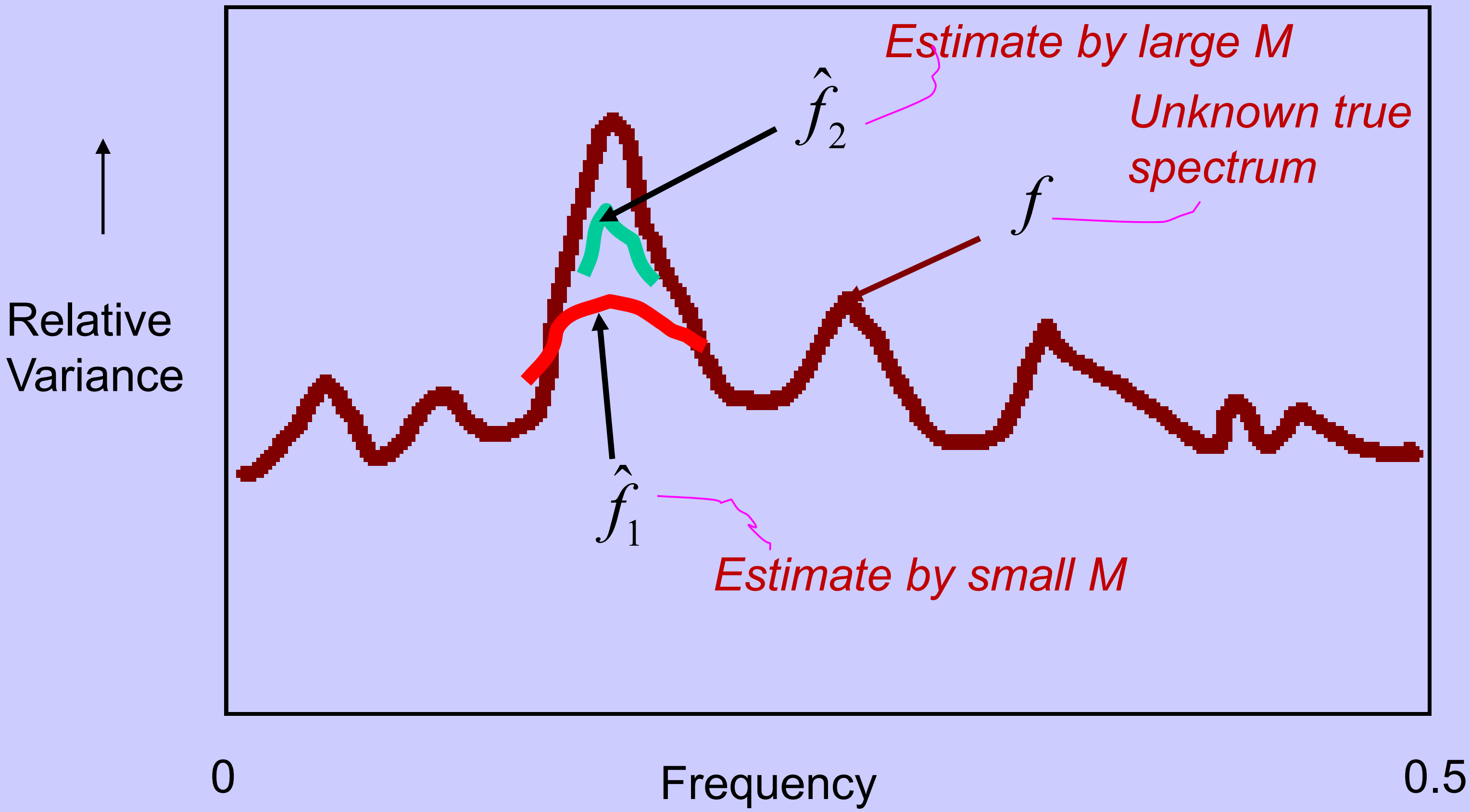
4. *Repeat 1-3 until satisfied that analysis adequately summarizes the important spectral features of the time series*

Smaller $M \rightarrow$ Increased bias

- Peaks and troughs “smoothed out”

- Can obscure important spectral features

Effect of choice of *M* on bias



Smaller M causes flattening of the spectral peak – estimate is biased low, and in the extreme gets smoothed out

Smaller *M* →smaller variance

- Confidence bands around spectral estimates narrower
- More likely to correctly identify an important spectral feature as "significant"
- Relationship

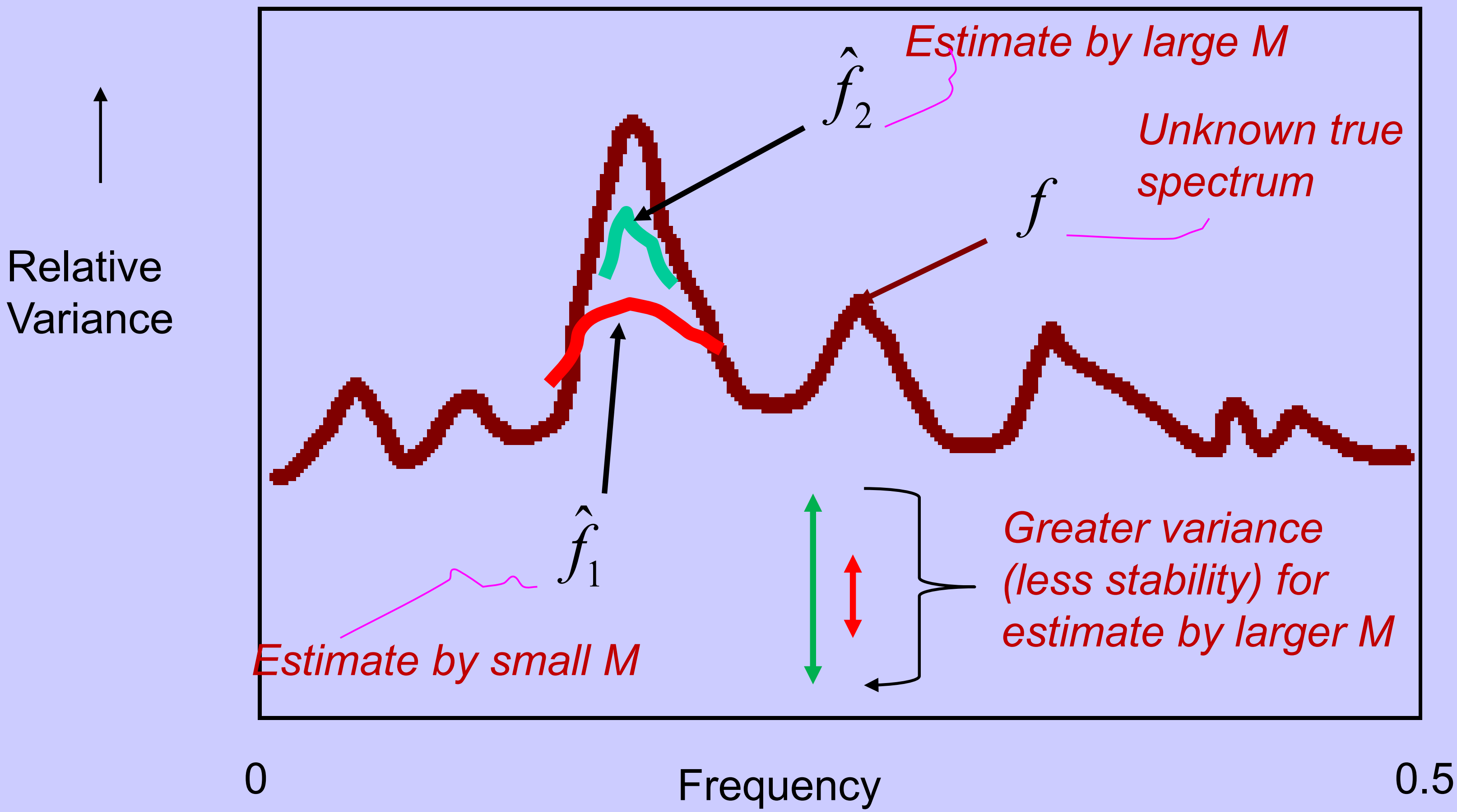
$$\nu \hat{f}(\omega) \big/ f(\omega) \quad \text{distributed as } \chi^2 \quad \text{with } \nu \text{ degrees of freedom}$$

Degrees of freedom depends on ratio

of sample size to truncation point:

$$\nu = 2.67 N \big/ M$$

Effect of choice of *M* on Variance

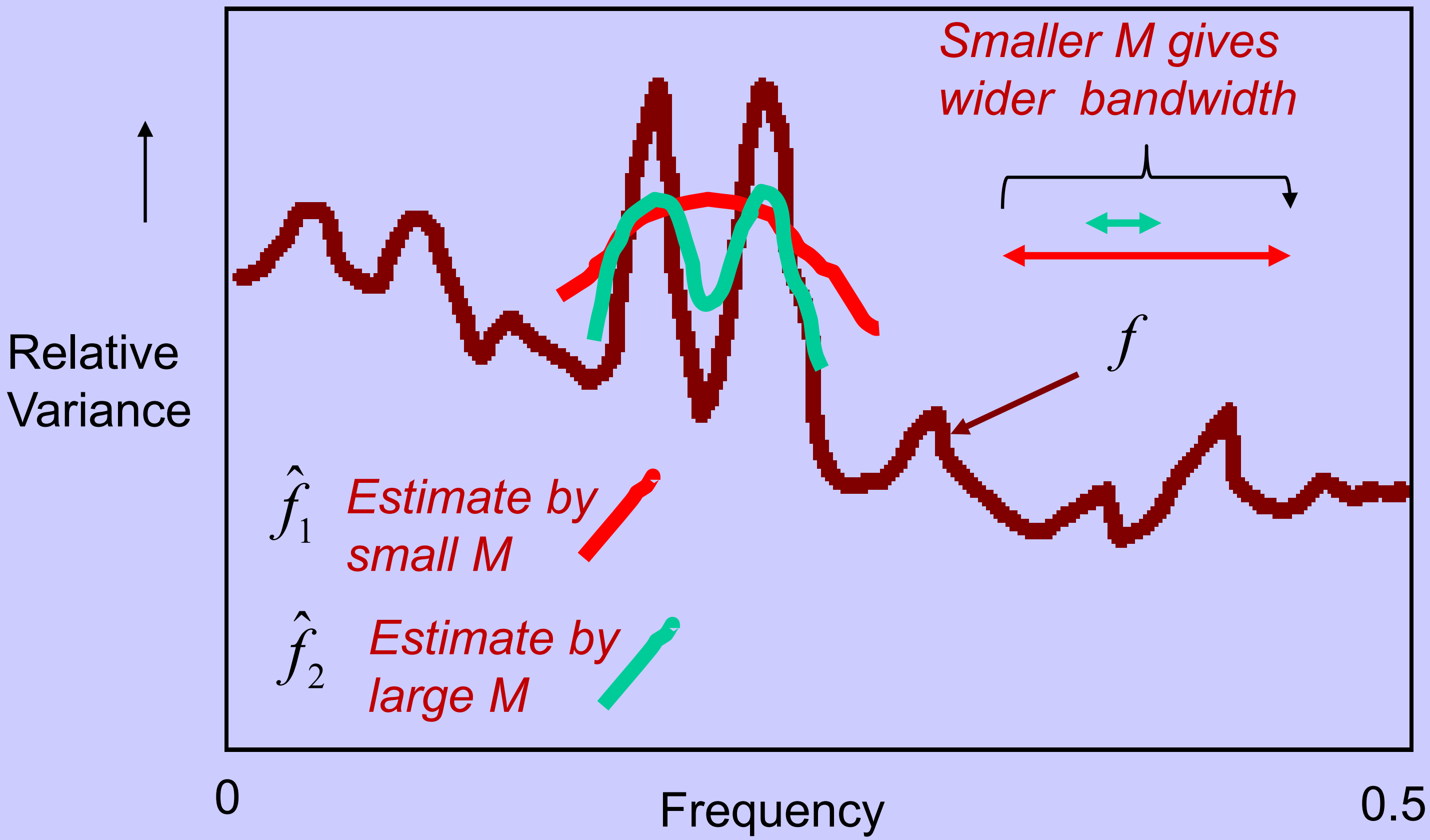


Smaller *M* → decreased resolution

- Averages over wider band of adjacent frequencies
- Adjacent spectral features tend to merge
- Effect summarized by widened bandwidth:

$$bw = 4/(3M) \quad \text{cycles/yr}$$

Effect of choice of *M* on resolution



Choice of M too small results in inability to distinguish peaks at closely spaced frequencies

Effect of choice of *M* on spectral estimate

M	**Bias**	**Variance**	**Resolution**
Small	High	Low	Low
Large	Low	High	High

Effect on variance is also described as effect on "stability" of the spectral estimate. Higher variance corresponds to LOWER variance.

Rough guide for setting *M*

- Start with $M \approx 2\sqrt{N}$, where N is the number of observations in time series

- Vary M: try 3 or 4 values, each time observing effect on spectrum

- One rule is that M should be in the range $\dfrac{N}{20}$ to $\dfrac{N}{3}$

  - *If M is small, the estimated spectrum may still show the main peaks, but is likely to be too smooth*

  - *If M is large, the spectrum will show many peaks, some of which may be spurious*

Trial runs of geosa4…